A brief introduction to characteristic classes from the differentiable viewpoint

Yang Zhang
LEPP, Cornell University

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1 Introduction

For the concepts of the associated fibre bundles, connection, curvature and the holomorphic vector bundles, please read the appendix.

2 Characteristic classes via the curvature forms

The local curvature form $\mathcal{F}$ on the base space for a fibre bundle should contain the information how the bundle is twisted. For example, for a trivial bundle, we can define a connection with the everywhere-vanishing local curvature form. To compare two bundles over the same base space M, we may try to simply compare the corresponding curvature forms $\mathcal{F}_1$ and $\mathcal{F}_2$ on X. However, the difficult is

- $\mathcal{F}$ is not globally defined on X. $\mathcal{F}$ defined on the patch U_i will differ from that defined on U_j by an adjoint action of the structure group G (gauge transformation). So it is hard to compare $\mathcal{F}_1$ and $\mathcal{F}_2$ by the local formula.

- For one fibre bundle, the compatible connection and thus the curvature form is not unique. So $\mathcal{F}$ contains redundant information about the fibre bundle.

So we will construct the *invariant polynomial* in terms of $\mathcal{F}$ which,

- is invariant under the adjoint action so it is not needed to think of the gauge transformation,
- can be reduced easily to get the connection-independent information of the fibre bundle.

2.1 Invariant polynomial

Recall that the local curvature form $\mathcal{F}$ is a Lie-algebra-valued (or Lie-algebra-representation-valued, for the vector bundle case) two-form. So before the discuss of invariant polynomial in terms of $\mathcal{F}$, we need to discuss the invariant polynomial of matrices.

Definition 2.1. If G is a Lie group with the Lie group $\mathfrak{g}$, a G-invariant r-linear symmetric function P is defined to be a map

$$P : \bigotimes_r \mathfrak{g} \to \mathbb{F} \tag{1}$$

such that,

1. For $c_1, c_2 \in \mathbb{C}$ and $A_i \in \mathfrak{g}$, $1 \leq i \leq r$,
$$\begin{aligned} &P(A_1, ...c_1 A_{i1} + c_2 A_{i2}, ..., A_r) \\ &= c_1 P(A_1, ..., A_{i1}, ..., A_r) + c_2 P(A_1, ..., A_{i2}, ..., A_r). \end{aligned} \tag{2}$$
2. For $1 \leq i, j \leq r$, $P(A_1, ...A_i, ..., A_j, ..., A_r) = P(A_1, ...A_j, ..., A_i, ..., A_r)$.
3. For $g \in G$, $P(Ad_g(A_1), ..., Ad_g(A_r)) = P(A_1, ..., A_r)$.

The set of all such P is denoted as $I^r(G)$.

Example 2.2. Let $G = GL(m, \mathbb{C})$ and consider its fundamental representation. For r $m \times m$ matrices A_i, $1 \leq i \leq r$, we define the *symmetric trace* as

$$str(A_1, ..., A_r) = \frac{1}{r!} \sum_{\sigma \in S^r} tr(A_{\sigma(1)} ... A_{\sigma(r)}) \tag{3}$$

where the sum is over all the permutations of $(1, ...r)$. It is clear that $str \in I^r(G)$.

We can combine all the $I^r(G)$ into a graded algebra $I^*(G) = \bigoplus_r I^r(G)$, where the product is defined to be: for $P \in I^p(G)$, $Q \in I^q(G)$, $A_i \in \mathfrak{g}$ and $1 \leq i \leq (p+q)$,

$$\begin{aligned} & (P \cdot Q)(A_1, ...A_{p+q}) \\ = & \frac{1}{(p+q)!} \sum_{\sigma \in S^{p+q}} P(A_{\sigma(1)}, ..., A_{\sigma(p)}) Q(A_{\sigma(p+1)}, A_{\sigma(p+q)}) \end{aligned} \tag{4}$$

where the sum is over all the permutations of $(1, ...p+q)$.

Definition 2.3. A *homogeneous invariant polynomial* P with the degree r is a map $P : \mathfrak{g} \to \mathbb{F}$, for which $\exists \tilde{P} \in I^r(G)$

$$P(A) = \tilde{P}(A, ..., A) \tag{5}$$

An *invariant polynomial* is the sum of finite homogeneous invariant polynomials with different degrees.

Example 2.4. If G has a k-dimensional representation, we can define

$$P(A) = \det(I + t\frac{iA}{2\pi}), \quad A \in \mathfrak{g} \tag{6}$$

where the determinant is over the $k \times k$ matrices. P is invariant under the adjoint representation. We can expend P in c,

$$P(A) = 1 + tP_1(A) + ... + t^k P_k(A) \tag{7}$$

then each $P_i(A)$ is also G-invariant and a degree-i homogeneous invariant polynomial. So P is an *invariant polynomial.* It is clear that $P_1(A) = \frac{i}{2\pi} tr(A)$ and $P_k(A) = \det(\frac{i}{2\pi}A)$. $\square$

Conversely, from a homogeneous invariant polynomial P we can find a $\tilde{P} \in I^r(G)$ which induces P

$$\tilde{P}(A_1, ..., A_r) = \frac{1}{r!} P(t_1 A_1 + ... + t_r A_r)\Big|_{t_1 ... t_r} \tag{8}$$

$\tilde{P}$ is called the *polarization* of P. As the previous example, we can check that the homogeneous invariant polynomial $P(A) = tr(A^r)$ has polarization $str(A_1, ..., A_r)$.

For $\tilde{P} \in I^r G$, We can extend its domain to r $\mathfrak{g}$-valued differential forms, as

$$\tilde{P}(A_1\eta_1, ...A_r\eta_r) \equiv \eta_1 \wedge ...\eta_r \tilde{P}(A_1, ...A_r), \tag{9}$$

and its linear extensions, where $A_i \in \mathfrak{g}$ and $\eta_i \in \Omega^{p_i}(M)$. Note that the η's may have different degrees. It is clear that,

$$\tilde{P}(Ad_g\Omega - 1, ...Ad_g\Omega_r) = \tilde{P}(\Omega - 1, ...\Omega_r) \tag{10}$$

for $g \in G$ and each Ω_i is a $\mathfrak{g}$-valued p_i-form.

Similarly, for a homogeneous invariant polynomial P with the degree r,

$$P(A\eta) = \big(\bigwedge_r \eta\big) P(A). \tag{11}$$

and its generalization for the non-homogeneous case is clear.

Consider the infinitesimal adjoint action $Ad_{\exp tX}$, $t \to 0$, we have,

Proposition 2.5. *for $X \in \mathfrak{g}$, $A_i \in g$ and $\tilde{P} \in I^r(G)$,*

$$\sum_{i=1}^{r} \tilde{P}(A_1, ..., [X, A_i], ..., A_r) = 0. \tag{12}$$

Furthermore, let A be a $\mathfrak{g}$-valued p-form and each Ω_i be a $\mathfrak{g}$-valued p_i-form.

$$\sum_{i=1}^{r} (-1)^{p(p_1+...+p_{i-1})} \tilde{P}(\Omega_1, ..., [A, \Omega_i], ..., \Omega_r) = 0. \tag{13}$$

Proof. The Lie algebra case (12) is self-evident. For (13), by the linearity of $\tilde{P}$, we just need to prove the case $A = X\eta$, where $X \in \mathfrak{g}$ and η is a p-form. By the definition (141), we have

$$[X, \Omega_i] = \eta \wedge [X, \Omega_i], \tag{14}$$

where $[X, \Omega_i]$ means the ad_X action on the Lie algebra component of Ω_i. So,

$$(-1)^{p(p_1+...+p_{i-1})} \tilde{P}(\Omega_1, ..., [A, \Omega_i], ..., \Omega_r) = \eta \wedge \tilde{P}(\Omega_1, ..., [X, \Omega_i], ..., \Omega_r). \tag{15}$$

Therefore (13) holds because of the infinitesimal case of (10). □

Proposition 2.6. *Let $\tilde{P} \in I^r(G)$ and each Ω_i be a $\mathfrak{g}$-valued p_i-form,*

$$d\tilde{P}(\Omega_1, ...\Omega_r) = \sum_{i=1}^{r} (-1)^{p_1+...+p_{i-1}} \tilde{P}(\Omega_1, ...d\Omega_i, ...\Omega_r). \tag{16}$$

2.2 Chern-Weil homomorphism

Let $\mathcal{F}$ be a local curvature form. On the intersection of two patches $U_i \cap U_j$,

$$\mathcal{F}_j = Ad_{t_{ij}^{-1}} F_i = t_{ij}^{-1} F_i t_{ij} \tag{17}$$

where the t_{ij} is the group element in G for the principal bundle case (183) or its V-representation for the vector bundle case (212). So $\mathcal{F}$ is not globally defined.

However, if P is a invariant polynomial, then $P(\mathcal{F})$ is globally defined.

Theorem 2.7 (Chern-Weil)**.** *$P(\mathcal{F})$ has the following good properties,*

1. *$dP(\mathcal{F})$=0.*

2. *If $\mathcal{F}$ and $\mathcal{F}'$ are the curvature forms of two connections of a fibre bundle respectively, then $P(\mathcal{F}_1) - P(\mathcal{F}_2)$ is exact.*

Proof. 1. It is sufficient to prove that case when P is a homogeneous invariant polynomial with the degree r. Let $\tilde{P}$ be the polarization of P,

$$\begin{aligned}
dP(\mathcal{F}) &= d\tilde{P}(\mathcal{F}, ..., \mathcal{F}) \\
&= \sum_{i=1}^{r} \tilde{P}(\mathcal{F}, ...d\mathcal{F}, ...\mathcal{F}) \\
&= \sum_{i=1}^{r} \tilde{P}(\mathcal{F}, ...d\mathcal{F}, ...\mathcal{F}) + P(\mathcal{F}, ...[\mathcal{A}, \mathcal{F}], ...\mathcal{F}) \\
&= \sum_{i=1}^{r} P(\mathcal{F}, ...\mathcal{D}\mathcal{F}, ...\mathcal{F}) = 0.
\end{aligned} \tag{18}$$

Because by the Bianchi identity $\mathcal{D}\mathcal{F} = d\mathcal{F} + [\mathcal{A}, \mathcal{F}] = 0$, where $\mathcal{A}$ is the local connection one-form. Here we used the propositions (13) and (16).

2. Let $\{(U_i, \phi_i)\}$ be the local trivialization of the fibre bundle. Then on each U_i, we have the connection one-forms $\mathcal{A}_i$ and $\mathcal{A}'_i$ from the two connections. We can define

$$\mathcal{A}_{it} \equiv \mathcal{A} + t\theta, \quad \theta = \mathcal{A}' - \mathcal{A}, \quad 0 \leq t \leq 1 \tag{19}$$

On the intersection $U_i \cap U_j$, $\mathcal{A}$ and $\mathcal{A}'$ satisfy the same transition law,

$$\mathcal{A}_j = Ad_{t_{ij}^{-1}}\mathcal{A}_i + t_{ij}^{-1}dt_{ij}, \quad \mathcal{A}'_j = Ad_{t_{ij}^{-1}}\mathcal{A}'_i + t_{ij}^{-1}dt_{ij} \tag{20}$$

So $\mathcal{A}_{it}$ and $\mathcal{A}_{jt}$ are compatible and we have a connection on the fibre bundle for each $0 \leq t \leq 1$. For the following calculate, we simply omit the subscript for patches. The curvature form for $\mathcal{A}_t$ is,

$$\mathcal{F}_t = d\mathcal{A}_t + \mathcal{A}_t \wedge \mathcal{A}_t = \mathcal{F} + t\mathcal{D}\theta + t^2\theta \wedge \theta \tag{21}$$

where $\mathcal{D}\theta = d\theta + [\mathcal{A}, \theta] = d\theta + \mathcal{A} \wedge \theta + \theta \wedge \mathcal{A}$. Hence,

$$\begin{aligned} P(\mathcal{F}') - P(\mathcal{F}) &= \int_0^1 dt \frac{d}{dt} P(\mathcal{F}_t) \\ &= r \int_0^1 dt \tilde{P}\Big(\frac{d}{dt}\mathcal{F}_t, \mathcal{F}_t ..., \mathcal{F}_t\Big) \\ &= r \int_0^1 dt \Big[\tilde{P}\Big(\mathcal{D}\theta, \mathcal{F}_t ..., \mathcal{F}_t\Big) + 2t\tilde{P}\Big(\theta \wedge \theta, \mathcal{F}_t ..., \mathcal{F}_t\Big)\Big] \end{aligned} \tag{22}$$

On the other hand,

$$\begin{aligned} d\tilde{P}(\theta, \mathcal{F}_t, ..., \mathcal{F}_t) &= \tilde{P}(d\theta, \mathcal{F}_t, ..., \mathcal{F}_t) - (r-1)\tilde{P}(\theta, d\mathcal{F}_t, ..., \mathcal{F}_t) \\ &= \tilde{P}(\mathcal{D}\theta, \mathcal{F}_t, ..., \mathcal{F}_t) - (r-1)\tilde{P}(\theta, \mathcal{D}\mathcal{F}_t, ..., \mathcal{F}_t) \end{aligned} \tag{23}$$

where we used the proposition (13). $\mathcal{D}_t\mathcal{F}$ is nonzero since $\mathcal{D}$ is defined by the connection form $\mathcal{A}$ not $\mathcal{A}_t$. So,

$$\mathcal{D}\mathcal{F}_t = \mathcal{D}_t\mathcal{F}_t - [t\theta, \mathcal{F}_t] = -t[\theta, \mathcal{F}_t]. \tag{24}$$

By using the proposition (13) again,

$$\begin{aligned} d\tilde{P}(\theta, \mathcal{F}_t, ..., \mathcal{F}_t) &= \tilde{P}(\mathcal{D}\theta, \mathcal{F}_t, ..., \mathcal{F}_t) + (r-1)t\tilde{P}(\theta, [\theta, \mathcal{F}_t], ..., \mathcal{F}_t) \\ &= \tilde{P}(\mathcal{D}\theta, \mathcal{F}_t, ..., \mathcal{F}_t) + 2t\tilde{P}(\theta \wedge \theta, \mathcal{F}_t, ..., \mathcal{F}_t) \end{aligned} \tag{25}$$

Hence,

$$\begin{aligned} P(\mathcal{F}') - P(\mathcal{F}) &= r \int_0^1 dt d\tilde{P}(\theta, \mathcal{F}_t, ..., \mathcal{F}_t) \\ &= rd\Big(\int_0^1 dt \tilde{P}(\theta, \mathcal{F}_t, ..., \mathcal{F}_t)\Big). \end{aligned} \tag{26}$$

Note that θ transform between different patches by the adjoint action of G.

□

Definition 2.8. For two connections on the same fibre bundle and an homogeneous invariant polynomial P_r, we define the *transgression* as,

$$TP_r = r\int_0^1 dt\tilde{P}(\mathcal{A}' - \mathcal{A}, \mathcal{F}_t, ..., \mathcal{F}_t). \tag{27}$$

TP_r is a globally defined $(2r-1)$-form on M.

Corollary 2.9. *Let the basis space M be a $2m$-dimensional orientable compact real manifold without boundary, P_m is a degree-m invariant polynomial, then*

$$\int_M P_m(\mathcal{F}) \tag{28}$$

is independent of the connection choice of the fibre bundle.

Proof. $\int_M P_m(\mathcal{F}') - \int_M P_m(\mathcal{F}) = \int_M d(TP_m) = 0$. □

Example 2.10. Let M be a two-dimensional compact real surface without boundary. If g is the Riemann metric of M, then we can locally choose the orthonormal frame $\{e_1, e_2\}$. Then local connection one-form is a $so(2)$-valued one form,

$$\begin{pmatrix} \mathcal{D}e_1 & \mathcal{D}e_2 \end{pmatrix} = \begin{pmatrix} e_1 & e_2 \end{pmatrix} \begin{pmatrix} 0 & \mathcal{A}^1{}_2 \\ \mathcal{A}^2{}_1 & 0 \end{pmatrix} \tag{29}$$

where $\mathcal{A}^1{}_2$ is a local one-form and $\mathcal{A}^2{}_1 = -\mathcal{A}^1{}_2$. Let $K = d\mathcal{A}^1{}_2$, then curvature matrix is

$$\begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix} \tag{30}$$

From differential geometry, we know that K is the Gaussian curvature multiplied by the volume form, so K depends on the metric. However, because $G = SO(2)$ is an abelian group, an element of the curvature matrix is G-invariant linear function. So by Chern-Weil theorem,

$$\int_m K \tag{31}$$

is independent of the metric choice. This is part of the classic *Gauss-Bonnet theorem*.

It is not easy to see this result if we do not use the orthonormal frame, because in general the curvature form is $gl(2, \mathbb{R})$-valued local curvature form. But in two-dimensional case, the only obvious invariant polynomials, $tr(\mathcal{F}) = R^{\rho}_{\rho\mu\nu} = 0$ and $\det(\mathcal{F}) = 0$ everywhere on M.

So given a fibre bundle E and invariant polynomial P, we can define a de Rham class $\chi_E(P) \equiv [P(\mathcal{F})] \in H^*(M)$. Chern-Weil theorem ensures that the change of the connection does not change the de Rham class. $\chi_E(P)$ is called a *characteristic class.*

Theorem 2.11. *1. $\chi_E : I^*(G) \to H^*(G)$ is a ring homomorphism. (Weil Homomorphism)*

2. Let $f : N \to M$ be a differentiable map and f^*E be the pullback fibre bundle of E. χ_E has the naturality property,

$$\chi_{f^*E}(P) = f^*\chi(E) \tag{32}$$

Proof. 1. It is clear that $\chi_E(\tilde{P}_1 + \tilde{P}_2) = \chi_E(\tilde{P}_1) + \chi_E(\tilde{P}_2)$ and $\chi_E(1) = [1] \in H^0(M)$, where $[1]$ is the class for the constant function $f(p) \equiv 1$, $\forall p \in M$. For the product, it is sufficient to prove the case when $\tilde{P}_r \in I^r(G)$ and $\tilde{P}_s \in I^s(G)$. Let $\mathcal{F} = \mathcal{F}^\alpha T_\alpha$.

$$\begin{aligned}(\tilde{P}_r\tilde{P}_s)(\mathcal{F}, ..., \mathcal{F}) &= \mathcal{F}^{\alpha_1} \wedge ... \wedge \mathcal{F}^{\alpha_r} \wedge \mathcal{F}^{\alpha_{r+1}} \wedge ... \wedge \mathcal{F}^{\alpha_{r+s}} \times \\ &\quad (\tilde{P}_r\tilde{P}_s)(T_{\alpha_1}, ..., T_{\alpha_r}, T_{\alpha_{r+1}}, ..., T_{\alpha_{r+s}}) \\ &= \mathcal{F}^{\alpha_1} \wedge ... \wedge \mathcal{F}^{\alpha_r} \wedge \mathcal{F}^{\alpha_{r+1}} \wedge ... \wedge \mathcal{F}^{\alpha_{r+s}} \times \\ &\quad \frac{1}{(r+s)!} \sum_{\sigma \in S_{r+s}} \tilde{P}_r(T_{\alpha_{\sigma(1)}}, ..., T_{\alpha_{\sigma(r)}}) \tilde{P}_s(T_{\alpha_{\sigma(r+1)}}, ..., T_{\alpha_{\sigma(r+s)}}) \\ &= \frac{1}{(r+s)!} \sum_{\sigma \in S_{r+s}} \mathcal{F}^{\alpha_{\sigma(1)}} \wedge ... \wedge \mathcal{F}^{\alpha_{\sigma(r)}} \wedge \mathcal{F}^{\alpha_{\sigma(r+1)}} \wedge ... \wedge \mathcal{F}^{\alpha_{\sigma(r+s)}} \times \\ &\quad \tilde{P}_r(T_{\alpha_{\sigma(1)}}, ..., T_{\alpha_{\sigma(r)}}) \tilde{P}_s(T_{\alpha_{\sigma(r+1)}}, ..., T_{\alpha_{\sigma(r+s)}}) = P_s(\mathcal{F}) \wedge P_r(\mathcal{F})\end{aligned}$$

where we used the fact that all $\mathcal{F}^\alpha$ commute with each other.

2. If on a local patch U_i, E has the local connection form $\mathcal{F}_i$, then on $f^{-1}(U_i)$, the curvature form is $f^*\Omega_i$.

□

Corollary 2.12. *For a trivial fibre bundle E, χ_E maps all the invariant polynomials to zero so all its characteristic classes are trivial.*

Proof. For a trivial fibre bundle, no matter it is a principal bundle or not, we can always choose a connection whose curvature form vanishes everywhere. Then $\chi_E(P) = [P(0)] = 0 \in H^*(M)$. $\square$

3 Chern Classes

3.1 Definition

From the example (2.4), for $A \in GL(k, \mathbb{C})$,

$$\det\left(I + t\frac{iA}{2\pi}\right) = \sum_{r=1}^{k} t^r P_r(A) \tag{33}$$

defined k degree-r invariant polynomials, P_r.

Definition 3.1. Let $\pi : E \to M$ be a complex vector bundle whose fibre is $\mathbb{C}^k$. Define the j-th *Chern Class* to be

$$c_j(E) = P_j(\mathcal{F}) \in H^{2j}(M) \tag{34}$$

and the *total Chern class* as

$$c(E) = c_0(E) + ... + c_k(E) \in H^*(M) \tag{35}$$

Proposition 3.2. *If a invariant polynomial $P_r \in I^r(GL(k, \mathbb{C}))$, can be written as a polynomial of matrix elements,*

$$P_r(A) = c_{\alpha_1\beta_1...\alpha_r\beta_r} A^{\alpha_1}{}_{\beta_1} ... A^{\alpha_r}{}_{\beta_r}, \forall A \in GL(k, \mathbb{C}) \tag{36}$$

then for a $gL(k, \mathbb{C})$-valued two-form $\mathcal{F}$, $P(\mathcal{F})$ equals the same polynomial in two-form matrix elements,

$$P_r(\mathcal{F}) = c_{\alpha_1\beta_1...\alpha_r\beta_r} \mathcal{F}^{\alpha_1}{}_{\beta_1} \wedge ... \wedge \mathcal{F}^{\alpha_r}{}_{\beta_r} \tag{37}$$

Proof. Define the matrix $E_{\alpha\beta}$ as the matrix with the element at (α, β) to be 1, and all the other elements vanished.

$$\begin{aligned}
P_r(\mathcal{F}) &= P_r(\mathcal{F}^{\alpha}{}_{\beta} E_{\alpha\beta}) = \tilde{P}_r(\mathcal{F}^{\alpha_1}{}_{\beta_1} E_{\alpha_1\beta_1}, ..., \mathcal{F}^{\alpha_r}{}_{\beta_r} E_{\alpha_r\beta_r}) \\
&= \mathcal{F}^{\alpha_1}{}_{\beta_1} \wedge ... \wedge \mathcal{F}^{\alpha_r}{}_{\beta_r} \tilde{P}_r(E_{\alpha_1\beta_1}, ..., E_{\alpha_r\beta_r}) \\
&= \mathcal{F}^{\alpha_1}{}_{\beta_1} \wedge ... \wedge \mathcal{F}^{\alpha_r}{}_{\beta_r} \frac{1}{r!} P_r(t_1 E_{\alpha_1\beta_1} + ... + t_r E_{\alpha_r\beta_r})\Big|_{t_1...t_r} \\
&= \mathcal{F}^{\alpha_1}{}_{\beta_1} \wedge ... \wedge \mathcal{F}^{\alpha_r}{}_{\beta_r} \frac{1}{r!}(c_{\alpha_1\beta_1...\alpha_r\beta_r} + \text{permutations}) \\
&= c_{\alpha_1\beta_1...\alpha_r\beta_r} \mathcal{F}^{\alpha_1}{}_{\beta_1} \wedge ... \wedge \mathcal{F}^{\alpha_r}{}_{\beta_r} \qquad (38)
\end{aligned}$$

where in the last line we used the fact that all the $\mathcal{F}^{\alpha}{}_{\beta}$ commute. □

By this proposition, $P_r(\mathcal{F})$ can be realized as the operators on the two-form-valued matrix. For example,

$$P_1(\mathcal{F}) = \frac{i}{2\pi} tr(\mathcal{F}) = \frac{i}{2\pi} \mathcal{F}^{\alpha}{}_{\alpha}, \qquad (39)$$

which is proportional to the Ricci form. And

$$P_k(\mathcal{F}) = \det\left(\frac{i\mathcal{F}}{2\pi}\right) \qquad (40)$$

where the multiplication between elements is the wedge product. Furthermore the total Chern class,

$$c(E) = [\det\left(I + \frac{i}{2\pi}\mathcal{F}\right)], \qquad (41)$$

and again, here the multiplication is the wedge product (or the product between a number and a form).

By the definition of de Rham cohomology,

$$c_j(E) = 0, \quad \text{if } 2j > \dim_{\mathbb{R}} M \qquad (42)$$

Proposition 3.3. *If the complex vector bundle E is equipped with a Hermitian metric h and its connection is Hermitian-compatible, then we can choose a real representative for each Chern class.*

Proof. From (247), if the Hermitian metric is $h = (h_{\alpha\bar{\beta}})$,

$$\overline{\mathcal{F}} = -h^{-1}\mathcal{F}^T h. \tag{43}$$

Hence,

$$\begin{aligned}\overline{\det\left(I + \frac{it}{2\pi}\mathcal{F}\right)} &= \det\left(I - \frac{it}{2\pi}\overline{\mathcal{F}}\right) = \det\left(I + \frac{it}{2\pi}h^{-1}\mathcal{F}^T h\right) &(44)\\ &= \det\left(I + \frac{it}{2\pi}\mathcal{F}\right). &(45)\end{aligned}$$

So the Chern classes are real forms. □

Example 3.4. Let $\pi : E \to M$ be a complex line bundle, i.e. the fibre is complex one-dimensional. Then the total Chern class is

$$c(E) = 1 + [\mathcal{F}], \tag{46}$$

where the curvature $\mathcal{F}$ is a real two-form.

Remark 3.5. Formally, we can use our "effective field theory" trick to calculate the Chern classes. For $A \in GL(m, \mathbb{C})$,

$$\begin{aligned}\det\left(I + \frac{i}{2\pi}A\right) &= \exp\left(\log\left(\det\left(I + \frac{i}{2\pi}A\right)\right)\right)\\ &= \exp\left(tr\left(\log\left(I + \frac{i}{2\pi}A\right)\right)\right)\\ &= \exp\left(-\sum_{n=1}^{\infty}\left(\frac{-i}{2\pi}\right)^n tr(A^n)\right)\\ &= 1 + \frac{i}{2\pi}tr(A) + \frac{1}{8\pi^2}(tr(A^2) - tr(A)^2) + \ldots, &(47)\end{aligned}$$

so it is generated by $tr(A^r)$. [1] Then by the proposition (3.2), the same formal works if A is replaced by $\mathcal{F}$.

Theorem 3.6. *Let $\pi_E : E \to M$ be a complex vector bundle.*

[1] This series can be calculated by Mathematica,

```
Series[Exp[-Sum[(-(I/(2 Pi)))^i t^i Array[tr,10][[i]]/i,{i,1,n}]],{t,0,n}]
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with n as the maximal degree of the invariant polynomial.

- *(Naturality) Let $f : N \to M$ be a smooth map. Then*

$$c(f^*E) = f^*c(E) \tag{48}$$

- *Let $\pi_F F \to m$ be another complex vector bundle. Then the total Chern class for the Whitney sum bundle $E \oplus F$ is,*

$$c(E \oplus F) = c(E) \wedge c(F) \tag{49}$$

Proof.

- It directly follows from the theorem (2.11).

- By the structure of the Whitney sum[2] , if E has the local trivialization $\{U_i, \phi_i\}$ and F has the local trivialization $\{V_l, \psi_l\}$, we can choose the local trivialization for $E \oplus F$ as,

$$\{U_i \cap V_l, \phi_i \oplus \psi_l\}, \tag{51}$$

  as long as $U_i \cap V_l \neq \emptyset$. If on U_i, the connection of E reads $\nabla s_\alpha = s_\beta \mathcal{A}^\beta{}_\alpha$, and similarly on V_l, F's connection reads $\nabla \sigma_a = \sigma_b A^b{}_a$, then the local connection form of $E \oplus F$ on $U_i \cap V_l$ for the sections $\{s_\alpha, \sigma_b\}$ is,

$$\begin{pmatrix} \mathcal{A}^\alpha{}_\beta & 0 \\ 0 & A^b{}_a \end{pmatrix} \tag{52}$$

  Hence, the local curvature form for $E \oplus F$ is,

$$\begin{pmatrix} \mathcal{F}^\alpha{}_\beta & 0 \\ 0 & F^b{}_a \end{pmatrix} \tag{53}$$

  therefore direct calculation shows that $c(E \oplus F) = c(E) \wedge c(F)$ via algebraic topology.

□

[2]Formally, the Whitney sum is defined as the follows,

$$\begin{tikzcd} & E \arrow[d, "\pi_E"] & \times & F \arrow[d, "\pi_F"] \\ X \arrow[r, "i"] & X & \times & X \end{tikzcd} \tag{50}$$

$E \times F$ is a complex vector bundle over $X \times X$. i is the diagonal map $p \to (p, p)$. Then $E \oplus F$ is defined to be the pullback $i^*(E \times F)$.

3.2 Chern Number

Chern classes were originally defined to be elements in $H^*(M, \mathbb{Z})$ via algebraic topology. Chern proved it is equivalent to Chern classes defined by curvature form, i.e. the j-th Chern classes $c_j(E) \in H^*(M, \mathbb{C})$ from the curvature form, determined $c'_j(E) \in H^*(M, \mathbb{Z})$ from algebraic topology, via

$$c'_j(E)(\sigma) \equiv \int_\sigma c_j(E) \tag{54}$$

for any $2j$-dimensional *singular cycle* σ in M with integer coefficients. It can be checked that, for the *universal bundle* the value of this integral is an integer. Then by the naturality of Chern classes,

$$\Delta_{2j} \xrightarrow{\sigma} M \xrightarrow{f} BG, \quad \int_\sigma c_j(E) = \int_{f\cdot\sigma} c_j(EG) \in \mathbb{Z} \tag{55}$$

so $c_j(E)' \in \mathrm{Hom}((H_{2j}M, \mathbb{Z}), \mathbb{Z})$. $\mathbb{Z}$ is PID, so $H^{2j}(M, \mathbb{Z}) = \mathrm{Hom}(H_{2j}(M, \mathbb{Z}), \mathbb{Z})$ and $c'_j(E) \in H^*(M, \mathbb{Z})$.

This equivalence implies that,

Proposition 3.7. *For any singular cycle σ in M with integer coefficients, the integral*

$$\int_\sigma c_{j_1}(E) \wedge ... \wedge c_{j_l}(E) \in \mathbb{Z}, \tag{56}$$

where if $\dim \sigma \neq j_1 + .. + j_l$*, the integral is simply set to zero.*

The integers obtained from all elements in M's singular homology are called *Chern Numbers*.

3.3 Splitting principle

From (49), if a complex vector bundle $pi : E \to M$ is the Whitney sum of several complex line bundles,

$$E = L_1 \oplus ... \oplus L_n \tag{57}$$

then

$$c(E) = c(L_1) \wedge ... \wedge c(L_n) = \prod_{i=1}^{n} (1 + x_i) \tag{58}$$

where x_i is the first Chern class for L_i. In this case, the matrix $\left(I + \frac{i\mathcal{F}}{2\pi}\right)$ is diagonalized.

However, in general $\left(I + \frac{i\mathcal{F}}{2\pi}\right)$ cannot be diagonalized or decomposed into the Jordan form, because it is not a complex-number matrix. Also E cannot be decomposed as the Whitney sum of n complex line bundles either. However, we have a strong claim that [5],

> (*The Splitting Principle*) To prove a polynomial identity in the Chern classes of complex vector bundles, it suffices to prove it under the assumption that the vector bundles are the Whitney sum of complex line bundles.

To see how to realize this principle, we need the following theorem,

Theorem 3.8. *Let $\pi : E \to M$ be a complex vector bundle of rank n. There exists a manifold $Fl(E)$, called the flag manifold associated with E and a smooth map $\xi : Fl \to M$ such that,*

- *the pullback of E to $Fl(E)$ splits into the Whitney sum of complex line bundles,*
$$\xi^{-1}(E) = L_1 \oplus ... \oplus L_n \tag{59}$$
- *$\xi^* : H^*(M) \to H^*(FL(E))$ is injective.*

Proof. See [5], section 21. □

Corollary 3.9. *In $H^*(Fl(E))$, by the naturality of Chern classes,*

$$\prod_{i=1}^{n} c(L_i) = \prod_{i=1}^{n} (1 + c_1(L_i)) = \xi^*(c(E)). \tag{60}$$

More generally, suppose that there are several complex vector bundles E_1, ... E_r over M. We can first introduce $\xi_1 : N_1 \to M$ which satisfies theorem (3.8), and $\xi_1^{-1}E_1$ is splitting on N_1. Then we can introduce $\xi_2 : N_2 \to M$ by theorem (3.8) and on N_2, $\xi_2^{-1}\xi_1^{-1}E_2$ is splitting. Note that $\xi_2^{-1}\xi_1^{-1}E_1$ is still splitting. Repeat this process and finally we get an $\xi : N \to M$, such that all $\xi^{-1}E_i$ are splitting and ξ^* is injective.

Now we see how this theorem implies that the *splitting principle*. For example, if we want to prove that for some polynomial P, complex vector bundles E and F over M,

$$P(c(E), c(F), c(E \otimes F)) = 0 \tag{61}$$

Choose $\xi : N \to M$ such that both $\xi^{-1}E$ and ξ^F are splitting and ξ^* is injective. The injective property means that it is sufficient to prove,

$$\xi^* P(c(E), c(F), c(E \otimes F)) = 0. \tag{62}$$

by the naturality of Chern classes, we only need to prove on N,

$$P(c(\xi^{-1}E), c(\xi^{-1}F), c(\xi^{-1}(E) \otimes \xi^{-1}(E)) = 0 \tag{63}$$

Now all the vector bundles are splitting, so it suffices to prove the splitting case only.

Proposition 3.10. *Let $P(x_1, ..., x_n)$ be a symmetric polynomial in $(x_1, ..., x_n)$,*

- *there exists a unique element $w_P(E) \in H^*(M)$, such that $P(c_1(L_1), ..., c_1(L_n)) = \xi^*(w)$.*
- *(Naturality). Let $f : N \to M$ be a smooth map. Then $w_P(f^{-1}E) = f^*(w_P(E))$.*

Proof.

- A symmetric polynomial is a polynomials in the elementary symmetric polynomials S_j,

$$P(x_1, ..., x_n) = Q(S_1(x_1, ..., x_n), ..., S_n(x_1, ..., x_n)) \tag{64}$$

Because $\prod_{i=1}^n (1 + c_1(L_i)) = \xi^*(c(E))$,

$$s_j(c_1(L_i), ...c_n(L_i)) = \xi^*(c_j(E)). \tag{65}$$

Therefore,

$$\begin{aligned} P(c_1(L_i), ...c_n(L_i)) &= Q(\xi^*(c_1(E)), ..., \xi^*(c_n(E))) \\ &= \xi^* Q(c_1(E), ...c_n(E)) \end{aligned} \tag{66}$$

Now we define $w_p(E) = Q(c_1(E), ...c_n(E))$. The uniqueness comes from the injective map ξ^*.

- By the naturality of Chern classes,

$$\begin{aligned} f^*w_p(E) &= f^*Q(c_1(E), ...c_n(E)) = Q(f^*c_1(E), ..., f^*c_1(E)) \\ &= Q(c_1(f^{-1}E), ..., c_1(f^{-1}E)) = w_p(f^{-1}E) \end{aligned} \tag{67}$$

Here we used the uniqueness of $w_p(f^{-1}E)$.

□

This proposition provides a simple way to construct characteristic classes in terms of Chern classes. In future, by abusing notations, we will denote such $w \in H^*(M)$ defined by P, as $w_P(E) \equiv P(x_1, ..., x_n)$, where x_i means the first Chern class of i-th complex line bundle on $Fl(M)$.

3.4 Chern character and Todd class

We can use Chern class to construct different characteristic classes,

Definition 3.11. The total Chern character is defined by

$$ch(E) \equiv [tr \exp\left(\frac{i\mathcal{F}}{2\pi}\right)] = \sum_{j=1} \frac{1}{j!}[tr\left(\frac{i\mathcal{F}}{2\pi}\right)^j] \tag{68}$$

and the $j-th$ Chern character is,

$$ch_j(E) = \frac{1}{j!}[tr\left(\frac{i\mathcal{F}}{2\pi}\right)^j] \tag{69}$$

Note that the exp series will truncate when $2j > dim_{\mathbb{R}}M$, so $ch(E)$ is a polynomial of $\mathcal{F}$ in de Rham cohomology.

Theorem 3.12. *Chern characters have the following properties,*

1. *(Naturality) For a smooth map $f : N \to M$, $ch(f^{-1}M) = f^*ch(M)$.*
2. $ch(F \oplus E) = ch(F) + ch(E)$.
3. $ch(F \otimes E) = ch(F) \wedge ch(E)$.

Proof. 1. The proof is exactly the same as that for the Chern classes.

2. We choose the local curvature form for $F \oplus E$ to be

$$\mathcal{F} = \begin{pmatrix} \mathcal{F}_F & 0 \\ 0 & \mathcal{F}_E \end{pmatrix} \tag{70}$$

so

$$tr \exp(\frac{i\mathcal{F}}{2\pi}) = tr \begin{pmatrix} \exp(\frac{i\mathcal{F}_F}{2\pi}) & 0 \\ 0 & \exp(\mathcal{F}_E) \end{pmatrix} = ch(F) + ch(E) \tag{71}$$

3. Locally, if that we have the frame $\{s_\alpha\}$ for F and $\{\sigma_a\}$ for E, then

$$\nabla_F \nabla_F s_\alpha = s_\beta \mathcal{F}^\beta_{F\alpha}, \quad \nabla_E \nabla_E \sigma_a = \sigma_a \mathcal{F}^a_{Fb}. \tag{72}$$

For the local frame $\{s_\alpha \otimes \sigma_a\}$, we have[3]

$$\begin{aligned} \nabla(s_\alpha \otimes \sigma_a) &= (\nabla_F s_\alpha) \otimes \sigma_a + s_\alpha \otimes (\nabla_E \sigma_a) && (74) \\ \nabla\nabla(s_\alpha \otimes \sigma_a) &= (s_\beta \mathcal{F}^\beta_{F\alpha}) \otimes \sigma_a + s_\alpha \otimes (s_b \mathcal{F}^b_{Ea}) && (75) \end{aligned}$$

The local curvature form for $F \otimes E$ is the Kronecker product $\mathcal{F}_F \otimes I + I \otimes \mathcal{F}_E$. Note that $\mathcal{F}_F \otimes I$ and $I \otimes \mathcal{F}_E$ commute, so

$$\begin{aligned} ch(E \otimes F) &= tr\Big(\exp(\frac{i}{2\pi}\mathcal{F}_F \otimes I) \exp(\frac{i}{2\pi} I \otimes \mathcal{F}_E) \Big) \\ &= tr\Big(\exp(\frac{i}{2\pi}\mathcal{F}_F) \otimes \exp(\frac{i}{2\pi}\mathcal{F}_E) \Big) \\ &= tr\Big(\exp(\frac{i}{2\pi}\mathcal{F}_F) \Big) \wedge tr\Big(\exp(\frac{i}{2\pi}\mathcal{F}_E) \Big) \\ &= ch(E) \wedge ch(F) && (76) \end{aligned}$$

□

Hence Chern character is a homomorphism from the ring of complex vector bundles to $H^*(M)$.

[3] We can extend the covariant derivative for the vector-valued forms as

$$\nabla(s \otimes \sigma) = (\nabla_F s) \otimes \sigma + (-1)^{pq} s \otimes (\nabla_E \sigma) \tag{73}$$

for local sections $s \in \Gamma(U_i, E) \otimes \Omega^p(U_i)$ and $\sigma \in \Gamma(U_i, F) \otimes \Omega^q(U_i)$

Recall that we can define the flag manifold of E and a map $\xi : Fl(E) \to M$, such that $\xi^{-1}E = L_1 \oplus ... \oplus L_n$. By the naturality of the Chern classes

$$\xi^*(c(E)) = \prod_{i=1}^{n}(1 + x_i), \quad x_i \equiv [\frac{i}{2\pi}F_i] = c_1(L_i), i = 1, ..., n \tag{77}$$

and $\xi^*(c_j(E))$ is the j-th elementary symmetric polynomial in $x_1, ..., x_n$.

For the Chern characters, again by the naturality,

$$\xi^*(ch(E)) = \sum_{i=1}^{n}[\exp(\frac{i}{2\pi}\mathcal{F}_i)] = \sum_{i=1}^{n} e^{x_i} \tag{78}$$

Comparing (77) and (78), because ξ^* is injective, we have,

$$\begin{gathered} ch_0(E) = n, \quad c_0(E) = 1 \\ ch_1(E) = c_1(E) \\ ch_2(E) = \frac{1}{2}c_1(E)^2 - c_2(E) \\ ... \end{gathered} \tag{79}$$

where by *Newton's identities*, we can rewrite all the Chern characters in terms of Chern classes.

Remark 3.13. We can prove the identity $ch(F \otimes E) = ch(F) \wedge ch(E)$ again by the splitting principle. Let $\xi : N \to M$ be the flag manifold such that $\xi^{-1}F = L_1 \oplus ... \oplus L_n$ and $\xi^{-1}F = l_1 \oplus ... \oplus l_k$. We have

$$\begin{aligned} \xi^*ch(F \otimes E) &= ch((\xi^{-1}F) \otimes (\xi^{-1}E)) = ch(\sum_{i=1}^{n}\sum_{j=1}^{k} L_i \otimes l_k) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{k} ch(L_i \otimes l_k) \end{aligned} \tag{80}$$

It is clear that for complex line bundles, $\mathcal{F}_{L_i \otimes l_k} = \mathcal{F}_{L_i} + \mathcal{F}_{l_k}$, so

$$ch(L_i \otimes l_k) = ch(L_i) \wedge ch(l_k) \tag{81}$$

Then,

$$\xi^*ch(F \otimes E) = \sum_{i=1}^{n} ch(L_i) \sum_{j=1}^{k} ch(l_k) \tag{82}$$

$$= \xi^*ch(F) \wedge \xi^*ch(E) = \xi^*(ch(F) \wedge ch(E)) \tag{83}$$

Because ξ^* is injective, this proved that $ch(F \otimes E) = ch(F) \wedge ch(E)$. In future, when we use the splitting principle, we can omit the notation ξ^* and simply assume that each complex vector bundle is splitting.

Definition 3.14. For a complex vector bundle $\pi : E \to M$, we define the *Todd class* to be the unique element $Td(E)$ in $H^*(M)$ such that,

$$\xi^*(Td(E)) = \prod_j \frac{x_j}{1 - e^{x_j}} \tag{84}$$

where again $\xi : N \to M$ is the flag manifold and $x_i = c_1(L_i)$

Because the product above is symmetry in all the x_j's. By the proposition (3.10), $Td(E)$ is well-defined and natural. Todd class has the following expansion,

$$\xi^*(Td(E)) = \prod_j \left(1 + \frac{1}{2}x_j + \sum_{k\geq 1}(-1)^{k-1}\frac{B_k}{(2k)!}x_j^{2k}\right), \tag{85}$$

where the B_k's are the *Bernoulli numbers*,

$$B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}, \quad B_3 = \frac{1}{42}, \quad B_4 = \frac{1}{30}, \quad B_5 = \frac{5}{66} \tag{86}$$

Theorem 3.15. *$Td(E \oplus F) = Td(E) \wedge Td(F)$.*

Proof. We use the splitting principle again. Here we omit ξ^* and simply put $E = L_1 \oplus ... \oplus L_n$ and $F = l_1 \oplus ... \oplus l_k$. Let $c_1(L_i) = x_i$ and $c_1(l_j) = x_{n+j}$

$$\begin{aligned} Td(E \oplus F) &= Td(L_1 \oplus ... \oplus L_n \oplus l_1 \oplus ... \oplus l_k) \\ &= \prod_{i=1}^{n+k} \frac{x_i}{1 - e^{x_i}} = \left(\prod_{i=1}^{n+k} \frac{x_i}{1 - e^{x_i}}\right) \wedge \left(\prod_{i=n+1}^{n+k} \frac{x_i}{1 - e^{x_i}}\right) \\ &= Td(E) \wedge Td(F). \end{aligned} \tag{87}$$

□

3.5 Application of Chern classes

Example 3.16 (line bundles on $\mathbb{C}P^1$). Here we show that several line bundles on $\mathbb{C}P^1$ are different. For line bundles, we just need to consider the first

Chern class c_1. $\mathbb{C}P^1$ is defined to be the one-dimensional complex projective space,

$$\{\mathbb{C}^2 - (0,0)\}/\{(z_0, z_1) \sim s(z_0, z_1), s \in \mathbb{C}^*\} \tag{88}$$

The local coordinates patches for $\mathbb{C}P^1$ are,

$$\begin{aligned} U_0 &= \{\mathbb{C}^2 - (0,0)|z_0 \neq 0\} \quad , \quad z \equiv z_1/z_0 \\ U_1 &= \{\mathbb{C}^2 - (0,0)|z_1 \neq 0\} \quad , \quad w \equiv z_0/z_1. \end{aligned} \tag{89}$$

On $U_0 \cap U_1$, the coordinates transit as $w = 1/z$. $\mathbb{C}P^1$ has the S^2 topology, we denote x to be the element in $H^2(\mathbb{C}P^1)$, such that

$$\int_{\mathbb{C}P^1} x = 1. \tag{90}$$

1. The trivial bundle $\mathbb{C}P^1 \times \mathbb{C}^1$. In this case, $c_1 = 0$.

2. The holomorphic tangle bundle $(T\mathbb{C}P^1)^+$. We know that $(TCP^1)^+$ allows the *Fubini-Study* metric, on U_0,

$$\frac{1}{(1+|z|^2)^2} dz \otimes d\bar{z}. \tag{91}$$

   On the intersection $U_0 \cap U_1$, the metric equals

$$\frac{1}{(1+|1/w|^2)^2} d\left(\frac{1}{w}\right) \otimes d\overline{\left(\frac{1}{w}\right)} = \frac{1}{(1+|w|^2)^2} dw \otimes d\bar{w} \tag{92}$$

   which is non-singular at $w = 0$. So the metric is globally defined. The curvature form on U_0 is,

$$\mathcal{F} = -\partial\bar{\partial} \log \left(\frac{1}{(1+|z|^2)^2} \right) = 2 \frac{dz \wedge d\bar{z}}{(1+|z|^2)^2} \tag{93}$$

   So the first Chern class is,

$$c_1 = \left[i \frac{dz \wedge d\bar{z}}{\pi(1+|z|^2)^2} \right]. \tag{94}$$

   The integral over $\mathbb{C}P^1$ gives,

$$\int_{\mathbb{C}P^1} c_1 = 2, \tag{95}$$

   so $c_1 = 2x$.

3. The holomorphic cotangent bundle $(T\mathbb{C}P^1)^{(1,0)}$. The metric for the cotangent bundle is just inverse of the metric for the tangent bundle. So repeat the calculation, we get $c_1 = -2x$.

4. The *canonical line bundle* γ^1, which is defined to be

$$\gamma^1 = \{[z_0, z_1] \otimes (u_0, u_1) | [z_0, z_1] \in \mathbb{C}P^1, (u_0, u_1) \in \mathbb{C}^2, [u_0, u_1] = [z_0, z_1]\} \tag{96}$$

and the bundle projection $\pi : \gamma^1 \to \mathbb{C}P^1$ is,

$$\pi([z_0, z_1] \otimes (u_0, u_1)) = [z_0, z_1]. \tag{97}$$

We have the following trivialization $\phi_0 : U_0 \times \mathbb{C} \to \pi^{-1}(U_0)$,

$$\phi_0(z, c) = [1, z] \otimes (c, cz) \tag{98}$$

and $\phi_0 : U_0 \times \mathbb{C} \to \pi^{-1}(U_0)$,

$$\phi_0(w, c') = [w, 1] \otimes (c'w, c') \tag{99}$$

The transition relation is $c' = zc$. We define the Hermitian metric on $\pi^{-1}(U_0)$ as,

$$(1 + |z|^2)c_1 c_2^* \tag{100}$$

while on $\pi^{-1}(U_0, U_1)$ the metric reads,

$$(1 + |z|^2)c_1 c_2^* = (1 + |z|^2)c_1' c_2'^* / |z|^2 = (1 + |w|^2)c_1' c_2'^* \tag{101}$$

which is nonsingular at $w = 0$. So this metric is globally defined. Repeat the calculation, we have $c_1 = -x$.

Hence all the line bundles have different first Chern classes and so they are different line bundles.

Example 3.17 (Dirac Monopole). Dirac Monopole is a point-like magnetic charge of the $U(1)$ gauge theory. Let the monopole has charge g, then $U(1)$ field strength is

$$F_{ij} = \epsilon_{ijk} \frac{g_m}{r^3} x^k, \quad r = \sqrt{x^2 + y^2 + z^2}. \tag{102}$$

We can think that the Dirac monopole configuration is a complex line bundle E over $\mathbb{R}^3 - (0,0,0)$. By the convention of Lie-algebra in physics, the curvature form [4] is $\mathcal{F} = ieF$ and,

$$c_1(E) = -\epsilon_{ijk}\frac{eg}{4\pi r^3}x^k dx^i \wedge dx^j. \tag{103}$$

Over a non-trivial singular cycle, $S^2 : r^2 = 1$, we have

$$\int_{S^2} c_1(E) = -2eg \tag{104}$$

By the proposition (3.7), the integral of a Chern class over a cycle must be an integer, otherwise it is not a well-defined bundle. So

$$2eg \in \mathbb{Z} \tag{105}$$

which is the Dirac quantization condition.

Example 3.18 (Instanton in 4D). Consider the $SU(2)$ Yang-Mills Theory in Euclidean $4D$ spacetime $\mathbb{R}^4$. We try to find field configurations corresponding to non-trivial fibre bundles. However, because R^4 is topologically trivial, it seems that no trivial fibre bundle can exist.

The point is that we are only interested in *finite action configuration*,

$$S_E = \frac{1}{2g^2}\int d^4x tr(F_{\mu\nu}F^{\mu\nu}) < \infty \tag{106}$$

which implies that,

$$F_{\mu\nu} \sim o(\frac{1}{r^2}), \quad r \to \infty. \tag{107}$$

So we may extend the fibre bundle to be over S^4. The instanton configurations correspond to non-trivial gauge bundles on S^4.

Let $SU(2)$ act on $\mathbb{C}^2$. We use the Euclidean convention such that $\mathcal{F} = F$, which *anti-Hermitian-matrix* valued. The first Chern classes vanished and the second one,

$$c_2(E) = \det\left(\frac{i\mathcal{F}}{2\pi}\right) = \frac{1}{8\pi^2}tr(F \wedge F), \tag{108}$$

[4] Note that in physics, we use $D_\mu = \partial_\mu + ieA_\mu$ instead of $D_\mu = \partial_\mu + \mathcal{A}_\mu$.

where we used the relation between Chern classes and Chern characters. Again, By the proposition (3.7),

$$\int_{S^4} \frac{1}{8\pi^2} tr(F \wedge F) = \int_{R^4} \frac{1}{8\pi^2} tr(F \wedge F) \in \mathbb{Z} \tag{109}$$

which is the instanton twisting number. Note that $c_2(E)$ is a closed form in S^4 but may not be exact. In R^4, we can find a three-form K, such that $dK = c_2(E)$ but K cannot be extend to the S^4.

Here we explicit construct the instanton solution, for the $R^4 - (0,0,0,0)$ patch of S^4, we can define a map,

$$g : \mathbb{R}^4 - (0,0,0,0) \to SU(2), \quad g(x_1, x_2, x_3, x_4) = \frac{x^4 + i\sigma \cdot \mathbf{x}}{r} \tag{110}$$

where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices. Let $w = g^{-1}dg$ be the Maurer-Cartan form on SU(2), define a pure-gauge configuration on $R^4 - (0,0,0,0)$ as,

$$\omega = g^* w. \tag{111}$$

which has zero field strength,

$$d\omega + \omega \wedge \omega = 0, \tag{112}$$

by the Maurer-Cartan equation. The field potential of an instanton would approach ω at $r \to \infty$, but vanishes at $(0,0,0,0)$ to smooth out the singularity of ω at the origin,

$$A = f\omega, \tag{113}$$

where f is a smooth function depends only on r. f vanishes at the origin and approach 1 for $r \to \infty$. We use a different approach to show the twisting is nontrivial:

The field strength is

$$F = df \wedge \omega + f d\omega + f^2 \omega \wedge \omega = df \wedge \omega + (f^2 - f)\omega \wedge \omega. \tag{114}$$

And the second Chern class is

$$\begin{aligned} c_2(E) &= \frac{1}{8\pi^2} tr(F \wedge F) \\ &= \frac{1}{8\pi^2} tr\Big(2(f^2 - f) df \wedge \omega \wedge \omega \wedge \omega + (f^2 - f)\omega \wedge \omega \wedge \omega \wedge \omega \Big) \\ &= \frac{1}{8\pi^2} tr\Big(2(f^2 - f) df \wedge \omega \wedge \omega \wedge w \Big) \end{aligned} \tag{115}$$

where $\omega \wedge \omega \wedge \omega \wedge \omega = g^*(w \wedge w \wedge w \wedge w) = 0$ because $\dim SU(2) = 2$. Using the Pauli matrix identities, we find that

$$tr(df \wedge \omega \wedge \omega \wedge \omega) = -12 \frac{df}{dr} \frac{1}{r^3} dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4. \qquad (116)$$

So the Chern number over S^4, or the instanton twisting number is

$$\begin{aligned}
\frac{1}{8\pi^2} \int_{S^4} tr(F \wedge F) &= \frac{1}{8\pi^2} \int_{R^4} tr(F \wedge F) \\
&= \frac{1}{8\pi^2} \int_0^\infty (2\pi^2) r^3 dr (-24)(f^2 - f) \frac{df}{dr} \frac{1}{r^3} \\
&= -6 \int_0^1 df (f^2 - f) = 1. \qquad (117)
\end{aligned}$$

which has no independence on the detail of f. If we demand the instanton has the *size* a, i.e.

$$f(r) = \frac{r^2}{r^2 + a^2}, \qquad (118)$$

then on the other patch of S^4, $(y^1, y^2, y^3, y^4) \equiv (x^1, x^2, x^3, x^4)/r^2$, we can check that $c_2(E)_{\mu\nu\lambda\rho}$ in the y-coordinates is finite at $(y^1, y^2, y^3, y^4) = (0, 0, 0, 0)$. So finite-size instanton configuration is corresponding to a non-singular and non-trivial fibre bundle.

Example 3.19 (D-brane action). In superstring theories, Dp-brane is a $p+1$ extend object which couples to fields in string theory supersymmetrically. The action for Dp-brane is rather complicated. Here we just consider the bosonic part of the action.

Type IIA or IIB string theory contains the gravitational field $G_{\mu\nu}$, the antisymmetric field $B_{\mu\nu}$, the dilaton Φ and the $U(N)$ gauge field $F_{\mu\nu}$, where N is the number of Dp-branes. These fields coupled to the brane via the *Dirac-Born-Infeld* action,

$$S_{DBI} = -\tau_p \int_{M_{p+1}} d^{p+1}\xi e^{-\Phi} tr(\sqrt{\det(g_{ab} + B_{ab} + 2\pi\alpha' F_{ab})}) \qquad (119)$$

where det acts on the Dp-branes coordinates indices while tr acts on the gauge indices. Because Dp-brane is a BPS state which breaks half of the supersymmetry, it must contain a BPS charge. So it seems that it would couple to RR fields of the superstring theory as,

$$S_{CS} = \mu_p \int_{M_{p+1}} C_{p+1} (?) \qquad (120)$$

However, it is not the whole story since the actually S_{CS} is much more complicated than this. For example, we consider one D-1 brane in the $1-2$ plane. Let $x^0 = \xi^0$, $x^1 = \xi^1$ and $x^2 = x^2(\xi^1)$ be the embedding of the D-1 brane. The Chern-Simons term reads,

$$S_{CS} = \mu_1 \int_{M_2} C_2 = \mu_1 \int d\xi^0 d\xi^1 \Big(C_{01} + \partial_1 X^2 C_{02} \Big) \tag{121}$$

If we apply a T-duality along x^2-direction, the D-1 brane becomes a D-2 brane, the X^2 coordinate becomes the gauge potential component A^2, so the action is now,

$$S_{CS} = \mu_2 \int d\xi^0 d\xi^1 d\xi^2 \Big(C_{012} + 2\pi\alpha' F_{12} C_0 \Big) = \mu_p \int_{M_{p+1}} \Big(C_3 + 2\pi\alpha' F C_1 \Big). \tag{122}$$

The Chern-Simon action is changed so (120) is not complete.

The complete form is

$$S_{CS} = \mu_p \int_{M_{p+1}} [\sum_j C_{j+1}] \wedge tr e^{2\pi\alpha' F + B}. \tag{123}$$

where $tre^{2\pi\alpha' F+B}$ is the Chern character for the mixture of F and B.

4 Pontryagin classes and Euler classes

Now we consider the real vector bundle E. Let $E^{\mathbb{C}} = E \oplus iE$ be its complexification.

Definition 4.1. We define the j-th and total Pontryagin class of E to be

$$p_j(E) = (-1)^j c_{2j}(E^{\mathbb{C}}) \in H^{4j}(M, \mathbb{Z}), \quad p(E) = \sum_j p_j(E) \tag{124}$$

The local frame $\{e_\alpha\}$ for E is also the local frame $E^{\mathbb{C}}$. Define that,

$$\nabla(ie_\alpha) = (ie_\beta)(\mathcal{A}^\beta)_\alpha. \tag{125}$$

So the local curvature form $(\mathcal{F}^\beta)_\alpha$ for $E^{\mathbb{C}}$ is the same as that for E. Then,

$$p(E) = \det\left(I + \frac{\mathcal{F}}{2\pi} \right)\Bigg|_{\text{even in } \mathcal{F}} \tag{126}$$

The new feature is, in cases we are interested in, $\mathcal{F}$ always comes from the Riemann structure g on E. In this case, $\mathcal{F}$ is Riemann anti-symmetric, (see the appendix),

$$\mathcal{F}^T = -gFg^{-1} \tag{127}$$

or $\mathcal{F}^T = -F$ if we use the orthonormal frame such that $g_{\alpha\beta} = \delta_{\alpha\beta}$. So

$$\det\left(I+\frac{\mathcal{F}}{2\pi}\right) = \det\left(I+\frac{\mathcal{F}^T}{2\pi}\right) = \det\left(g(I+\frac{\mathcal{F}}{2\pi})g^{-1}\right) = \det\left(I-\frac{\mathcal{F}}{2\pi}\right) \tag{128}$$

so the odd terms in $\mathcal{F}$ automatically vanished and we can define

$$p(E) \equiv \det\left(I + \frac{\mathcal{F}}{2\pi}\right). \tag{129}$$

The first several Pontryagin classes are,

$$p_1(E) \;=\; -\frac{1}{8\pi^2}tr(\mathcal{F}^2) \tag{130}$$

$$p_2(E) \;=\; \frac{1}{128\pi^4}[(tr\mathcal{F}^2)^2 - 2tr(\mathcal{F}^4)] \tag{131}$$

Example 4.2. For a Riemann manifold M. The local curvature form is

$$(\mathcal{F}^\mu)_\nu = \frac{1}{2}R^\mu{}_{\nu\lambda\rho}dx^\lambda \wedge dx^\rho \tag{132}$$

so the first Pontryagin class is,

$$p_1(TM) = -\frac{1}{32\pi^2}R^\mu{}_{\nu\lambda^1\lambda^2}R^\nu{}_{\mu\lambda^3\lambda^4}dx^{\lambda_1} \wedge dx^{\lambda_2} \wedge dx^{\lambda_3} \wedge dx^{\lambda_4} \tag{133}$$

Consider a real $2n \times 2n$ antisymmetric matrix A. We have

$$det(A) = Pf(A)^2 \tag{134}$$

where $Pf(A)$ is the *Pfaffian* of A,

$$Pf(A) = \frac{(-1)^l}{2^l l!}\sum_\sigma sgn(\sigma)A_{\sigma_1\sigma_2}...A_{\sigma_{2n-1}\sigma_{2n}}. \tag{135}$$

Definition 4.3. For a $2n$-dimensional orientable real manifold. Choose the orthonormal frames consistent with the orientation, we define the *Euler class* as

$$e(M) = Pf\left(\frac{\mathcal{F}}{2\pi}\right) \tag{136}$$

A Lie-algebra-valued differential forms

Let M be a differential manifold and V is a linear space. A V-valued n-form ϕ on M is defined to be an element in $\Omega^n(M) \otimes V$. ϕ at a point $p \in M$ naturally induced a linear map,

$$\phi_p : T_pM \wedge ... \wedge T_pM \to V. \tag{137}$$

This map can be interpreted as the intrinsic definition of the a V-valued n-form. In particular, if $V = \mathfrak{g}$ is a Lie algebra, then ϕ is called a Lie-algebra-valued n-form. Furthermore, if $\mathfrak{g}$ is the Lie algebra of G, $Ad_g\phi = g\phi g^{-1}$ is the adjoint action of $g \in G$ on the Lie-algebra component of ϕ.

The exterior derivative on $\Omega(M) \otimes V$ is,

$$d(\sum_i \eta_i \otimes v_i) \equiv d\eta_i \otimes v_i, \tag{138}$$

where $\eta_i \in \Omega^n(M)$ and $v_i \in V$.

If V itself is a $\mathbb{R}$ or $\mathbb{C}$ associate algebra, i.e., V has a product $\cdot$ structure which is linear under $\mathbb{R}$ or $\mathbb{C}$, we can define the exterior product of two V-valued forms as,

$$(\eta \otimes v) \wedge (\eta' \otimes v') = (\eta \wedge \eta') \otimes (v \cdot v') \tag{139}$$

and its linear extensions. Here $\eta \in \Omega^n(M)$, $\eta' \in \Omega^m(M)$ and $v, v' \in V$. In particular, when $V = \mathfrak{g}$, the product $\cdot$ is defined by the product in the *universal enveloping algebra* of $\mathfrak{g}$, i.e. the product of the matrices, not the commutator.

Note that the usual commutation relation,

$$\phi \wedge \phi' = (-1)^{mn}\phi' \wedge \phi \tag{140}$$

does not hold for $\phi \in \Omega^m(M) \otimes V$ and $\phi' \in \Omega^n(M) \otimes V$. Hence We define the commuatator of them as

$$[\phi, \phi'] = \phi \wedge \phi' - (-1)^{mn}\phi' \wedge \phi. \tag{141}$$

and it is still a Lie-algebra-valued form.

For $\mathfrak{g}$-valued m-from, $\phi = T_i\eta_i$ and n-forms, $\phi' = T_i\eta'_i$

$$[\phi, \phi'] = [T_i, T_j] \otimes (\eta_i \wedge \eta'_j) \tag{142}$$

and in particular,

$$[\phi, \phi] = \begin{cases} 2\phi \wedge \phi = [T_i, T_j] \otimes (\eta_i \wedge \eta_j) & m \text{ is odd} \\ 0 & m \text{ is even} \end{cases} \tag{143}$$

Let G be a Lie group and $\mathfrak{g}$ is its Lie algebra.

Definition A.1. The *Maurer-Cartan form* of G is the unique $\mathfrak{g}$-valued one-form w such that for a vector X in T_gG,

$$w(X) = (L_{g^{-1}})_* X \tag{144}$$

It is clearly the w is invariant under L_g^*. If G is embedded in $GL(n, R)$ and its element g is written as a $n \times n$ matrices, then explicitly its Maurer-Cartan form is $g^{-1}dg$. Conventionally, we may use $g^{-1}dg$ as the general notation of the Maurer-Cartan form.

Recall that for a one-form w, two vector fields X and Y,

$$d\omega(X, Y) = X[\omega(Y)] - Y[\omega(X)] - \omega([X, Y]) \tag{145}$$

This relation also holds for V-valued one form. Let ω to be the Maurer-Cartan form and X and Y to be *left-invariant* vector fields. At any point $g \in G$, we have $w(Y|_g) = Y_e$ which is independent of g, so $X[w(Y)] = 0$. Hence,

$$\begin{aligned} dw([X|_g, Y|_g]) &= -w([X, Y]|_g) = -[X, Y]|_e \\ &= -[w(X|_g), w(Y|_g)] = -(w \wedge w)(X|_g, Y|_g), \end{aligned} \tag{146}$$

where in the first we use the fact that the commutator of two left-invariant fields is still left-invariant. Since the directions of $X|_g$ and $Y|_g$ are arbitrary, we have

Theorem A.2 (Maurer-Cartan equation)**.** $dw + w \wedge w = dw + \frac{1}{2}[w, w] = 0$.

It is useful to rewrite the Maurer-Cartan equation as the component form. Let $\{T_i\}$ be in $\mathfrak{g}$, we can decompose the Maurer-Cartan form as,

$$w = T_i \otimes w^i, \tag{147}$$

where w^i's are one-forms on G. It is clear that w^i's are also left invariant. Let $\tilde{T}_i$ be the corresponding left-invariant vector fields of T_i, and the *structure constants* are,

$$[\tilde{T}_i, \tilde{T}_j] = f_{ij}^k \tilde{T}_k \tag{148}$$

We have,

$$w^i(\tilde{T}_j) = \delta^i_j, \tag{149}$$

and Theorem A.2 becomes,

$$dw^k + \frac{1}{2} f^k_{ij} w^i \wedge w^j = 0. \tag{150}$$

The Maurer-Cartan equation plays a crucial role in the study of the fiber bundle curvature.

B Connection in principal bundles

B.1 Right action and the vertical space

This is a short review of properties of connections in fiber bundles. The notations follow [1]. Let $P(M,G)$ be a principal bundle with the total space P, base space M and the structure group G. G acts on P naturally by the right action,

$$\begin{array}{cccccc} R: & P & \times & G & \to & P \\ & u & & g & \mapsto & ug. \end{array} \tag{151}$$

The infinitesimal right action linearly maps a Lie algebra element $A \in \mathfrak{g}$ to a vector field $A^\#$ in the total space P: Let $f_u : g \mapsto ug$ be the right action restricted at the point $u \in P$, then $A^\#$ at u is defined to be,

$$A^\#\Big|_u = f_{u*} A. \tag{152}$$

It is clear that $A^\#\Big|_u$ is in V_uP, the vertical subspace. Since f_{u*} is injective, $\# : \mathfrak{g} \to V_uP$ is an isomorphism. $A^\#$ is a smooth vector field on P and the flow generated on P is $u \exp(At)$.

Proposition B.1. *For $A, B \in \mathfrak{g}$, $[A^\#, B^\#] = [A,B]^\#$.*

Proof.

$$\begin{aligned} [A^\#, B^\#] &= L_{A^\#} B^\# = \lim_{t \to 0} \frac{1}{t} \Big(R_{\exp(-tA)*} B^\# - B^\# \Big) \\ &= \lim_{t \to 0} \frac{1}{t} \Big(R_{\exp(-tA)*} \circ L_{\exp(tA)*} B - B \Big)^\# = [A,B]^\# \end{aligned} \tag{153}$$

where in the second line, we use the fact that for $u \in P$ and $g_1, g_2 \in G$, $(ug_1)g_2 = u(g_1 g_2)$. $\square$

Let T_i be the basis of $\mathfrak{g}$, then at each point $u \in P$, $T_i^\#$ is a basis of $V_u P$. If a vertical vector field X on P, i.e., X is vertical everywhere, then $X = a^i T_i^\#$ and a^i are smooth functions in P. For two vertical vector fields $X = a^i T_i^\#$, $Y = b^i T_i^\#$,

$$[X, Y] = [a^i T_i^\#, b^j T_j^\#] = a^i b^j [T_i, T_j]^\# + a^i (T_i^\# b^j) T_j^\# - b^j (T_j^\# a^i) T_i^\#. \quad (154)$$

where (B.1) is used. Each term is vertical, therefore,

Proposition B.2. *For two vertical vector fields X and Y, $[X, Y]$ is vertical.*

B.2 Connection

In general, it is not straightforward to globally and smoothly define the horizontal subspace, $H_u P$, the complement of $V_u P$ in $T_u P$ for each $u \in P$. So we need to introduce the connection.

Definition B.3. A *connection* on P is a separation of the tangent space at each point of P, into the vertical and horizontal space, $T_u P = V_u P \oplus H_u P$, $\forall u$, such that,

- C1 (Smoothness) A smooth vector field X on P is separated as $X = X^V + X^H$, such that $X^V \in V_u P$, $X^H \in H_u P$ and both X^V and X^H are smooth.
- C2 (Right invariance) $H_{ug} P = R_{g*} H_u P$ $\forall u \in P$ and $g \in G$.

The second property ensures that a horizontal lift of a curve is still horizontal under the right group action.

Proposition B.4. *For $A \in \mathfrak{g}$ and a horizontal vector field Y, $[A^\#, Y]$ is vertical.* [5]

[5] Nakahara's Lemma 10.2 [1] is not correct. Here is the correct version from [2]. Nakahara's lemma claimed that for $X \in H_u P$ and $Y \in V_u P$, $[X, Y] \in H_u P$. If it is true, then for a smooth function ϕ on P, ϕX is still vertical but $[\phi X, Y] = \phi[X, Y] - (Y\phi)X$ whose second term is not horizontal. The problem in Nakahara's proof is that although there always is a $A\mathfrak{g}$ such that $A^\#|_u = X$, $A^\#$ may *not* equal X everywhere.

Proof. $A^\#$ generate the right action, so

$$[A^\#, Y] = L_{A^\#}Y = \lim_{t\to 0} \frac{1}{t}(R_{\exp(-At)*}Y - Y). \tag{155}$$

Since the right action R_{g*} maps the horizontal space into another horizontal space, both of the two terms are horizontal. □

In practice, it is convenient to introduce the *connection one-form* to specify a connection on a principal bundle.

Definition B.5. A connection one-form is a Lie-algebra-valued one form $\omega \in \mathfrak{g} \otimes T^*P$ such that,

CF1 $\omega(A^\#) = A$, $\forall A \in g$.

CF2 $R_g^*\omega = Ad_{g^{-1}}\omega$, where $Ad_{g^{-1}}$ is the adjoint action on the Lie-algebra component of the connection one-form.

Then we define the H_uP to be,

$$H_uP \equiv \{X \in T_uP|\omega(X) = 0\}. \tag{156}$$

It is clear that (CF1) guaranteed that $H_uP \cap V_uP = 0$ and (CF2) guaranteed that the right invariance (C2) for the connection is satisfied. The connection one-form was strictly defined by Ehresmann.

B.3 Local connection form

The connection one-form ω is globally defined on the total space P. For the simplicity, it would be convenient to define the "projection" of ω on the base space M. Unfortunately, the global projection on M does not exist in general. We need to define the projections on the open covering of M and find the transition formula.

Let U_i be an open covering of M and σ_i be the local section defined on U_i. The canonical local trivialization on U_i is $\phi_i : U_i \times G \to \pi^{-1}(U_i), (p, g) \mapsto \sigma_i(p)g$.

$$\begin{tikzcd} U_i \cap U_j \times G \arrow[rr, "\phi_i^{-1}\phi_j"] \arrow[dr, "\phi_j"'] & & U_i \cap U_j \times G \arrow[dl, "\phi_i"] \\ & \pi^{-1}(U_i \cap U_j) & \end{tikzcd} \tag{157}$$

where $\phi_i^{-1}\phi_j(p,g) = (p, t_{ij}(p)g)$. Here $t_{ij}(p) \in G$ is the transition function and $\sigma_j(p) = \sigma_i(p)t_{ij}(p)$.

Remark B.6. It is a good place to see why it is necessary to use the connection to define the horizontal space. Can we naively define the horizontal space H_uP as $(\phi_j)_*T_pM$? No. The problem is that this definition is not consistent with the transition function. It is clear that for $u = \phi_j(p,g)$, $X \in T_pM$,

$$(\phi_j)_*(X, 0) = (\phi_i)_*(X, (R_g t_{ij})_* X). \tag{158}$$

Hence in general, $(\phi_j)_*T_pM \neq (\phi_i)_*T_pM$ and H_uP cannot be defined in this way.

We define the *local connection one-form* $\mathcal{A}_i$ on U_i as,

$$\mathcal{A}_i \equiv \sigma_i^* \omega \tag{159}$$

Ont the other hand, give a $\mathfrak{g}$-valued one-form on A_i on U_i, we can define the corresponding connection one-form ω_i on $\pi^{-1}(U)$, as

$$\omega_i = Ad_{g_i^{-1}}(\pi^* \mathcal{A}_i) + g_i^{-1} dg_i \tag{160}$$

where g_i is the local coordinate for the total space such that $\phi(p,g) = u$. So $g_i : \pi^{-1}(U) \to G$ is a well-defined mapping and $g_i^{-1}dg_i$ is the pullback of the Maurer-Cartan form of G. It is straightforward to verify that,

- $\sigma_i^* \omega_i = \mathcal{A}_i$.
- ω_i is a connection one-form on the bundle $\pi^{-1}(U_i)$, i.e., ω_i satisfies the conditions (CF1) and (CF2).

The *compatibility condition* for the local connection one-forms is $\omega_i = \omega_j$ in $U_i \cap U_j, \forall i, j$. By the explicit construction (160), the compatibility condition can be determined as,

$$\mathcal{A}_j = t_{ij}^{-1} \mathcal{A}_i t_{ij} + t_{ij}^{-1} dt_{ij}. \tag{161}$$

This relation is the gauge transformation in Yang-Mills theory.

A nontrivial fiber bundle does not have a global section, so there are several patches U_i and each patch has a local connection one-forms $\mathcal{A}_i$. Since on each patch the bundle is trivial, A_i itself does not contain the global information of the bundle. It is the transformation (161) that determines the global structure.

B.4 Horizontal lift and the holonomy group

For a curve $\gamma(t)$ in M, it is clear that there exist many curves $\tilde{\gamma}(t)$ in P such that $\pi(\tilde{\gamma}(t)) = \gamma(t)$. We call $\tilde{\gamma}(t)$ a lift of $\gamma(t)$. In general $\tilde{\gamma}(t)$ can have "vertical-direction-motion", We want to find a unique lift which is always in the horizontal direction,

Definition B.7. For a curve $\gamma : [0,1] \to M$, its lift $\tilde{\gamma} : [0,1] \to P$ is γ's horizontal lift if and only if $\tilde{\gamma}_*(\partial/\partial t)$ is horizontal $\forall t \in [0,1]$.

We have the following theorem for the existence and uniqueness of horizontal lift,

Theorem B.8. *Let $\gamma : [0,1] \to M$ be a C^1 (continuously differentiable) curve in M and $u \in \pi^{-1}(\gamma(0))$. Then there exists a unique continuously differentiable curve $\tilde{\gamma}$ such that $\pi(\tilde{\gamma}) = \gamma$ and $\tilde{\gamma}(0) = u$.*

Proof. For the proof without using the local sections, see [2]'s proposition 3.1. Here we sketch the proof with the local sections. Let U_α, σ_α to be the local trivializations of M. By the compactness of $[0,1]$, we can divide the interval as $[t_0 = 0, t_1], ..., [t_{N-1}, t_N]$ such that each segment of the curve $\gamma : [t_{n-1}, t_n] \to M$ is inside a U_n. For $t_0 \le t \le t_1$, we need to construct a horizontal lift as $\tilde{\gamma}(t) = \sigma_1(\gamma(t))g_1(t)$ in U_1, where $\sigma_1(\gamma(0))g_1(0) = u$. The derivative of $\tilde{\gamma}$ is,

$$\frac{d\tilde{\gamma}}{dt} = R_{g_1*}\left(\sigma_{1*}(\frac{d\gamma}{dt})\right) + \left((Lg_1)_*^{-1}\frac{dg_1}{dt}\right)^{\#}, \tag{162}$$

By the horizontal condition,

$$0 = \omega(\frac{d\tilde{\gamma}}{dt}) = Ad_{g_1^{-1}}\left(\omega\big(\sigma_{1*}(\frac{d\gamma}{dt})\big)\right) + (Lg_1)_*^{-1}\frac{dg_1}{dt} \tag{163}$$

which is a differential equation for $g_1(t)$,

$$\frac{dg_1}{dt} = -\omega\left(\sigma_{1*}(\frac{d\gamma}{dt})\right)g_1 = -\mathcal{A}_1\left(\frac{d\gamma}{dt}\right)g_1, \tag{164}$$

where in the second equality we used the local connection form $\mathcal{A}_1 = \sigma_1^*\omega$. By the fundamental theorem of ODE's, the horizontal lift exist uniquely for $t \in [0, t_1]$. Repeat this process for $N-1$ times, we get the unique horizontal lift for $t \in [0,1]$. $\square$

Formally, in each patch, the horizontal lift can be formally written as the *path-ordered* form, i.e. to put late-time operator before the early-time operator in the product.

$$g_n(t) = \mathcal{P}\exp\Big(- \int_{t_{i-1}}^{t} \mathcal{A}_n(\frac{d\gamma}{dt})dt\Big) g_n(t_{n-1}) \tag{165}$$

$$= \mathcal{P}\exp\Big(- \int_{t_{i-1}}^{t} A_{n,i}\frac{dx^i}{dt}dt\Big) g_n(t_{n-1}) \tag{166}$$

where we used the local coordinate for U_n in the second line. $g_n(t_{n-1})$ is determined by the previous patch.

For a curve $\gamma(t)$ in M and its horizontal lift $\tilde{\gamma}(t)$ with $\tilde{\gamma}(0) = u$, we denote the $\tilde{\gamma}(1)$ as the *parallel transport* of u along γ. The parallel transport induces an isomorphism Γ_γ between $\pi^{-1}(\gamma(0))$ and $\pi^{-1}(\gamma(1))$ because of the uniqueness of the horizontal lift.

Because of the right-invariance of the connection form (CF2), we have the following proposition:

Proposition B.9. *If $\tilde{\gamma}(t)$ is the horizontal lift of $\gamma(t)$ with $\tilde{\gamma}(0) = u$, then If $\tilde{\gamma}(t)g$ is the horizontal lift of $\gamma(t)$ with $\tilde{\gamma}(0) = ug$,*

In the other word, if $v \in P$ is the parallel transport of u along γ, then vg is the parallel transport of ug along γ.

Now we consider the closed C^1 loop on M, $\gamma(0) = \gamma(1) = \pi(u)$. The parallel transport for u along γ is in the same fibre of u, so it can be written as ug_γ, where $g_\gamma \in G$ is the unique right action determined by γ. Consider all the C^1 loops passing $\pi(u)$, we define the *holonomy group* at u as,

$$\Phi_u = \{g_\gamma | \gamma(t) \text{ is a } C^1 \text{ curve in } M, \gamma(0) = \gamma(1) = \pi(u)\}. \tag{167}$$

Φ_u is a subgroup of G because,

$$g_{\gamma_1} g_{\gamma_2} = g_{\gamma_2 * \gamma_1}, \quad g_\gamma^{-1} = g_{\gamma^{-1}}. \tag{168}$$

Note the order of the product of the two loops. Similarly, we can define the *restricted holonomy group* as,

$$\Phi_u^0 = \{g_\gamma | \gamma(t) \text{ is a } C^1 \text{ curve in } M, \gamma(0) = \gamma(1) = \pi(u), [\gamma] = id. \in \pi(M,u)\}. \tag{169}$$

Proposition B.10. *The holonomy groups has the following properties,*

1. *For $u \in P$, $\Phi_{ug} = Ad_{g^{-1}}(\Phi_u)$ and $\Phi^0_{ug} = Ad_{g^{-1}}(\Phi^0_u)$.*

2. *If $u, v \in P$ can be connected by a horizontal line, then $\Phi_u = \Phi_v$ and $\Phi^0_u = \Phi^0_v$.*

Proof. 1. Let $v = ua$ be the parallel transport of u along γ. By (B.9), $vg = uag = ug(g^{-1}ag)$ is the parallel transport of ug along γ. So $Ad_{g^{-1}}$ is an isomorphism from Φ_u to Φ_{ug} (and for the restrict holonomy group).

2. Let $\alpha(t)$ to be the horizontal curve from u to v. If a loop γ, starting and ending at $\pi(v)$, induces the parallel transport from v to vg_γ, then the loop $\alpha * \tilde{\gamma} * R_{g_\gamma}(\alpha^{-1})$ is horizontal curve starts from u and ends at ug_γ. This proved $\Phi_u = \Phi_v$. Furthermore if γ is trivial in $\pi(M, \pi(v))$, the projection of $\alpha * \tilde{\gamma} * R_{g_\gamma}(\alpha^{-1})$ is trivial in $\pi(M, \pi(u)$. So $\Phi^0_u = \Phi^0_v$.

So for a connected base space we just need the holonomy group for an arbitrary point $u \in P$.

□

B.5 Curvature form

The connection on a principal bundle $P(M, G)$ separates $T_uP = H_uP \oplus V_uP$: $X = X^H + X^V$. Let V be a linear space, for a V-valued n-form ϕ on P, we define the *covariant derivative* of ϕ by,

$$D\phi(X_1, ..., X_{r+1}) = d\phi(X^H_1, ..., X^H_{r+1}). \tag{170}$$

So $D\phi$ is a V-valued $(n+1)$-form. Because under R_{g*}, $H_uP \to H_{ug}P$ and $V_uP \to V_{ug}P$, we have $(R_{g*}X)^H = R_{g*}(X^H)$. Then it is straightforward to check that,

$$R^*_g \circ D = D \circ R^*_g. \tag{171}$$

The *curvature two-form* Ω is a $\mathfrak{g}$-valued two form defined as,

$$\Omega \equiv D\omega. \tag{172}$$

Because of (171) and (CF2),

$$R^*_g\Omega = Ad_{g^{-1}}\Omega. \tag{173}$$

Theorem B.11 (Cartan structure equation)**.**

$$\Omega = d\omega + \omega \wedge \omega = d\omega + \frac{1}{2}[\omega, \omega] \tag{174}$$

Proof. It is sufficient to prove that for any $X, Y \in T_u P$.

$$\Omega(X, Y) \equiv d\omega(X^H, Y^H) = d\omega(X, Y) + [\omega(X), \omega(Y)] \tag{175}$$

Every X in $T_u P$ can be decomposed as $X = X^H + X^V$, so by the linearity, we only need to consider three cases,

- $X \in H_u P$ and $Y \in H_u P$. By the definition of the horizontal space, $\omega(X) = \omega(Y) = 0$. So $\Omega(X, Y) = d\omega(X, Y)$.
- $X \in V_u P$ and $Y \in V_u P$. In this case, by definition the left hand side of (175) is 0. Let $A, B \in \mathfrak{g}$ such that $A^\#|_u = X$ and $B^\#|_u = Y$. Then, [6]

$$\begin{aligned} d\omega(X, Y) &= A^\#(\omega(B^\#)) - B^\#(\omega(A^\#)) - \omega[A^\#, B^\#] \\ &= A^\#(B) - B^\#(A) - [A, B] = [A, B] \end{aligned} \tag{176}$$

  where in the second line we used the proposition B.1. Hence $d\omega(X, Y) + [\omega(X), \omega(Y)] = [A, B] - [A, B] = 0$ which equals the left hand side.
- $X \in V_u P$ and $Y \in H_u P$. In this case, again the left hand side of (175) is zero. Define $A\mathfrak{g}$ such that $A^\#|_u = X$. We also extend Y to a horizontal vector field, which is still called Y. Then,

$$d\omega(X, Y) = -Y(\omega(A^\#)) - \omega([A^\#, Y]) = 0, \tag{177}$$

  because $[A^\#, Y]$ is horizontal by proposition B.4. It is clear that $[\omega(X), \omega(Y)] = 0$. So again, both sides of (175) are zero.

$\square$

Example B.12. For each local trivialization (U_i, σ_i) such that $u = \sigma_i(\pi(u))g$, $\forall u \in \pi^{-1}(U_i))$, we can define the *canonical flat connection* $\omega_i = g^{-1}dg = g^* w$, where w is the Maurer-Cartan form of G. It is clear that the horizontal space

[6] This formula holds for any extension of the vectors X, Y. So it does not matter if $A^\# \neq X$ outside the point u.

defined by ω_i is $\sigma_{i*}T_PM$ for $p = \pi(u)$. Therefore $\mathcal{A}_i = \sigma_i^*\omega_i = 0$. w_i has zero curvature, because

$$\Omega_i = d\omega_i + \omega_i \wedge \omega_i = g^*(dw + w \wedge w) = 0, \tag{178}$$

where we used the Maurer-Cartan equation (A.2). For a principal bundle P with the global connection form ω, if we can find a set of local trivializations (U_i, σ_i) such that $\omega = \omega_i$ for all U_i, ω is called the *flat connection*. Such ω has zero curvature everywhere.

Theorem B.13 (Bianchi's identity). $D\Omega = 0$

Proof. For vectors $X, Y, Z \in T_uP$, we have

$$D\Omega(X, Y, Z) = d\Omega(X^H, Y^H, Z^H) = (d\omega \wedge \omega - \omega \wedge d\omega)(X^H, Y^H, Z^H), \tag{179}$$

which vanishes since no matter which horizontal vector is combined with ω, the result is always zero. Here we used the Cartan structure equation. □

Definition B.14. We define the *local curvature form* on a local trivialization (U_i, σ_i) as, $\mathcal{F} = \sigma_*\Omega$.

Pull back the Cartan structure equation (B.11) by σ_i^*, we get its local form

$$\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} \tag{180}$$

which is the *field strength* expression in Yang-Mills theory.

Example B.15 (Pure gauge). The local connection $\mathcal{A}$ on U_i is called *pure gauge* if $\mathcal{A} = g^{-1}dg$, where $g : U_i \to G$ is a differentiable map. Pure guage connection has zero local curvature. Let w to be the Maurer-Cartan form on G. Then $\mathcal{A} = g^*w$. By the Maurer-Cartan equation (A.2), $dw + w \wedge w = 0$ so $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = g^*(dw + w \wedge w) = 0$.

Proposition B.16. *The local form of the Bianchi's identity is,*

$$0 = d\mathcal{F} + [\mathcal{A}, \mathcal{F}] \equiv \mathcal{D}\mathcal{F} = 0. \tag{181}$$

Proof. By Cartan structure equation (B.11), $d\Omega = d\omega \wedge \omega - \omega \wedge d\omega$. The pullback by σ_i^* is

$$0 = d\mathcal{F} - d\mathcal{A} \wedge A + \mathcal{A} \wedge d\mathcal{A} = d\mathcal{F} + [\mathcal{A}, \mathcal{F}]. \tag{182}$$

□

On the intersection of two branches U_i and U_j, the local curvature forms are related by,

$$\mathcal{F}_j = Ad_{t_{ij}^{-1}} F_i. \tag{183}$$

which can be checked by the direct calculation. This is the gauge transformation for field strength in Yang-Mills theory.

B.6 Ambrose-Singer Theorem

The non-triviality of the horizontal lift for a principal bundle P is determined by the curvature form Ω of P. Ambrose and Singer proved the holonomy theorem for the relation between the restricted holonomy ground and the curvature form.

Theorem B.17 (Ambrose-Singer)**.** *For $u \in P$, the Lie algebra of Φ_u is spanned by $\{\Omega(X,Y)|X,Y \in H_vP\}$ for all $v \in P$ such that v can be connected with u by a horizontal curve.*

One important conclusion from the holonomy theorem is that:

Proposition B.18. *For a principal bundle $P(M,G)$ with the connection form ω and the curvature form Ω,*

- *The connection of $P(M,G)$ is flat if and only if Ω vanishes everywhere.*
- *If Ω vanishes every where and M is simply connected, then $P(M,G)$ is isomorphic to the trivial bundle $M \times G$ and by this isomorphism ω is mapped to the canonical flat connection form of $M \times G$.*

Proof. See [2], theorem 9.1. □

C Connection on associated fibre bundles

C.1 Associated fibre bundles

Definition C.1. Let the structure group G act on a manifold V,

$$G \times V \to V \tag{184}$$

then from a principal bundle $P(M,G)$ we can construct its *associated bundle* $E(M,V,G,P)$ with the fibre V as,

$$E(M,V,G,P) = P \times_G V = P \times V/\{(u,v) \sim (ug, g^{-1}v)\}. \tag{185}$$

The fibre bundle projection is,

$$\pi_E : E \to M, \quad [(u, v)] \to \pi(u). \tag{186}$$

In the following discussion we may simply use the notation π instead of π_E.

The local trivialization of E is naturally determined by the P's local sections (U_i, σ_i). Define the local trivialization of E on U_i as $\phi_i : U_i \times V \to \pi^{-1}(U_i) : (p, v) \mapsto [(\sigma_i(p), v)]$. The transition relation is now,

$$\begin{tikzcd}
U_i \cap U_j \times V \arrow[rr, "\phi_i^{-1}\phi_j"] \arrow[dr, "\phi_j"'] & & U_i \cap U_j \times V \arrow[dl, "\phi_i"] \\
& \pi^{-1}(U_i \cap U_j) &
\end{tikzcd} \tag{187}$$

where $\phi_i^{-1}\phi_j(p, v) = (p, t_{ij}(p)v)$. Recall that $\sigma_j(p) = \sigma_i(p)t_{ij}(p)$. So for the associated bundle, the transition function is the V-representation of the transition function of the principal bundle.

Although G acts on V, it does not naturally act on $\pi_E^{-1}(p)$, for p in M. We define the *left action referred* to u as, where $u \in \pi_P^{-1}(p)$,

$$L_u(g)([ua, v]) \equiv [(u, gav)], \quad a \in G, g \in G \tag{188}$$

L_u is a group homomorphism: $G \to Aut(\pi_E^{-1}(p))$. Note that $L_{ua}(g) = L_u(a)L_u(g)L_u(a^{-1})$.

C.2 Connection on associated fibre bundle

For the principal bundle $P(M, P)$ with a connection, it is natural to define the corresponding connection of the associated bundle $E(M, V, G, P)$.

Definition C.2 (Connection)**.** For $w = [(u, v)] \in E$, we define the *horizontal space* $H_wE \subset T_wE$ to be

$$H_wE = \left\{ \frac{d}{dt}[(\tilde{\gamma}(t), v)]_{t=0} | \tilde{\gamma}(t) \text{ is horizontal in } P, \tilde{\gamma}(0) = u \right\} \tag{189}$$

This definition has no dependence of the particular choice (u, v) either. We can use another representative $w = [(ug, g^{-1}v]$, $a \in G$ and a horizontal curve $\tilde{\gamma}$ with $\tilde{\gamma}(0) = ug$. Then $[(\tilde{\gamma}(t), g^{-1}v)] = [(\tilde{\gamma}(t)g^{-1}, v)]$ and $\tilde{\gamma}(t)g^{-1}$

is also horizontal in P because of (C2). By the local trivialization, it is straightforward to check that

$$T_wE = H_wE \oplus V_wE. \tag{190}$$

and $\pi_E : H_wE \to T_pM$ is an isomorphism.

Again, a curve $\tilde{\gamma}_E(t)$ in E is a *horizontal lift* of a curve $\gamma(t)$ in M, if and only if $\pi_E(\tilde{\gamma}_E(t)) = \gamma(t)$ and $d/dt(\tilde{\gamma}_E(t))$ is horizontal in E, $\forall t$. If the horizontal lift of $\gamma(t)$ is known, $\tilde{\gamma}_E(t)$ can be determined as follows: Let $\gamma(0) = p$, $\pi_E(w) = p$, $w = [(u, v)]$, $\tilde{\gamma}(t)$ be the horizontally lift of $\gamma(t)$ in P. Then,

$$\tilde{\gamma}_E(t) = [(\tilde{\gamma}(t), v)] \tag{191}$$

is the horizontal lift of $\gamma(t)$ in E with the starting point w. Again, the horizontal lift in the associated bundle with specified starting point exists and is unique.

Let γ be a smooth loop in M such that $\gamma(0) = \gamma(1) = p$. Again, The horizontal lift on E evaluated at $t = 1$ gives a linear isomorphism,

$$\gamma : \pi_E^{-1}(p) \to \pi_E^{-1}(p), \tag{192}$$

which is called the *parallel transport* along γ. As before, consider all the smooth loops or contractible loops, we define the *holonomy group* Hol_p and *restricted holonomy group* Hol_p^0, as the subgroup of $Aut(\pi^{-1}(p))$.

Proposition C.3. $\mathrm{Hol}_p = L_u(\Phi_u)$ *and* $\mathrm{Hol}_p^0 = L_u(\Phi_u^0)$, *where* Φ_u *and* Φ_u^0 *are the holonomy groups at* u *for the principal bundle* P.

Proof. For a loop γ in M such that $\gamma(0) = \gamma(1) = p$. Let $\tilde{\gamma}$ be the lift of γ in P with $\tilde{\gamma}(0) = u$ and $\tilde{\gamma}(1) = ug$. Then $[(\tilde{\gamma}(t), v)]$ is a horizontal lift of γ in E. The parallel transport of any $[(u, v)] \in \pi_E^{-1}(p)$ is,

$$[(\tilde{\gamma}(1), v)] = [(ug, v)] = [(u, gv)] = L_u(g)([(u, v)]). \tag{193}$$

Therefore for any element in $T \in \mathrm{Hol}_p$, there exists an element $g \in G$ such that $L_u(g) = T$. □

Note that $\Phi_{ua} = Ad_{a^{-1}}(\Phi_u)$, so

$$L_{ua}(\Phi_{ua}) = L_u(Ad_a(\Phi_{ua})) = L_u(\Phi_u), \tag{194}$$

which is consistent with the fact that Hol_p is independent of the choice of u.

C.3 Covariant derivative

In this section, we assume the V is a vector space with the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. The linear operators on $E(M, V, P, G)$ are

- For $w_1, w_2 \in \pi_E^{-1}(p)$, choose $w_1 = [(u, v_1)]$ and $w_2 = [(u, v_2)]$ and $w_1 + w_2 = [(u, v_1 + v_2)] \in \pi_E^{-1}(p)$.
- For $w \in E$ and $c \in \mathbb{F}$, choose $w = [(u, v)]$ and $cw = [(u, cv)]$.

For a vector bundle, the parallel transformation is linear, so Hol_p and Hol_p^0 are in $Gl(\pi^{-1}p, \mathbb{F})$.

And we can naturally identify a vertical vector $A \in V_w E$ defined by,

$$A = \frac{d}{dt}[(u, v(t))]\Big|_{t=0}, \quad v : \mathbb{R} \to V \tag{195}$$

as an element of E in the same fibre of w,

$$A \equiv [(u, \dot{v}(0))]. \tag{196}$$

Let $s \in \Gamma(M, E)$ be a differentiable section on E, X be a vector $T_p M$. Denote $w = s(p)$. In general, the pushforward of X by s, $s_* X$, may not be horizontal in $T_w E$. We define the *covariant derivate*, $\nabla_X s$, to indicate the discrepancy between s and the horizontal lift of X,

Definition C.4 (Covariant derivative)**.**

$$\nabla_X s = (s_* X)^V. \tag{197}$$

$\nabla_X s$ is vertical and can be identified as an element in $\pi_E^{-1}(p)$. This definition does not refer to the particular representatives of E.

Explicitly, $\nabla_X s$ can be calculated as follows: extend X into a curve γ, $\gamma(0) = p$ with $\dot{\gamma}(0) = X$. Let $w = (u, v)$ and denote $\tilde{\gamma}(t)$ as the horizontal lift of $\gamma(t)$ in P with $\tilde{\gamma}(0) = u$. Then $\tilde{\gamma}_E(t) = [(\tilde{\gamma}(t), v)]$ is the horizontal lift in E. If $s(\gamma(t)) = [(\gamma(t), \eta(t))]$, then

$$s_* X = \frac{d}{dt}\tilde{\gamma}_E(t)\bigg|_{t=0} + [(\gamma(0), \dot{\eta}(0))] \tag{198}$$

where the second term is the vertical component. Hence,

$$\nabla_X s = [(\gamma(0), \dot{\eta}(0))]. \tag{199}$$

This formula implies that $\nabla_X s$ is determined only by the values of s along $\gamma(t)$. Furthermore, if $\nabla_{\dot{\gamma}(t)} X = 0$, $\forall t$, then $s(\gamma(t))$ is the horizontal lift of $\gamma(t)$ in E.

For a smooth vector field X, $\nabla_X s$ is a global section of E with $(\nabla_X s)(p) = \nabla_{X_p} s$. Formally, we define ∇s to be an element of $\Gamma(M, E) \otimes \Omega^1(M)$ such that at each $p \in M$

$$X(\nabla s) \equiv \nabla_X s, \quad \forall X \in T_p M. \tag{200}$$

It straightforward to check that the covariant derivatives satisfy,

CD1 $\nabla(c_1 s_1 + c_2 s_2) = c_1 \nabla s_1 + c_2 \nabla s_2$,

CD2 $\nabla(fs) = (df)s + f\nabla s$,

for $c_1, c_2 \in \mathbb{F}$ and $f \in C^\infty(M)$. Alternatively we can define the covariant derivate to be a map $\nabla : \Gamma(M, E) \to \Gamma(M, E) \otimes \Omega^1(M)$ which satisfies axioms (CD1) and (CD2). In this way we can recover the whole theory of the connection.

C.4 Local form of the connection

Let ω be the connection form on $P(M, G)$. It is not straightforward to write down the corresponding global connection form on $E(M, V, P, G)$. So we try to find the local connection form of $E(M, V, P, G)$ in terms of the local connection form of $P(M, G)$.

As before, (U_i, σ_i) is the set of local trivializations of $P(M, G)$. The local connection form of P on U_i is $\mathcal{A}_i = \sigma_i^* \omega$. Choose a basis $\{e_\alpha\}$ for V. The action $G \times V \to V$ induces the Lie-algebra representation $\mathfrak{g} \times V \to V$,

$$T e_\alpha = T^\beta{}_\alpha e_\beta, \quad T \in \mathfrak{g} \tag{201}$$

where $T^\beta{}_\alpha$ is the V-representation matrix. Note the $\{e_\alpha\}$ is a basis not the components, so the matrix $T^\beta{}_\alpha$'s first index contract with e_β.

On U_i, we can define the local *canonical frame* as,

$$s_\alpha(p) = [(\sigma_i(p), e_\alpha)]. \tag{202}$$

It is clear that at each point $p \in U_i$, these sections form a basis for $\pi_E^{-1}(p)$. $\phi_i : U_i \times V \to \pi^{-1}(U_i)$, where $\phi_i(p, v) = [(\sigma_i(p), v)]$ is the *canonical trivializations* of E.

Proposition C.5. $\nabla s_\alpha = (\mathcal{A}_i)^\beta{}_\alpha s_\beta$.

Proof. Let X be a vector in T_pM and $\gamma(t)$ is a curve in M such that $\dot\gamma(0) = X$. Define $\tilde\gamma(t) = \sigma_i(p(t))g_i(t)$ as the horizontal lift of $\gamma(t)$, and $g_i(0) = e \in G$. Then the horizontal lift of $\gamma(t)$ in E with the starting point $s_\alpha(p)$ is,

$$\tilde\gamma_E(t) = [(\sigma_i(p(t))g_i(t), e_\alpha)] \tag{203}$$

while $s_i(p(t))$ reads,

$$s_i(p(t)) = [(\sigma_i(p(t)), e_\alpha)] = [(\sigma_i(p(t))g_i(t), g_i(t)^{-1}e_\alpha)]. \tag{204}$$

Therefore,

$$\begin{aligned}
\nabla_X e_\alpha &= \left[\left(\sigma_i(p), \frac{d}{dt}\left(g_i(t)^{-1}e_\alpha\right)_{t=0}\right)\right] \\
&= \left[\left(\sigma_i(p), -g_i(t)^{-1}\frac{d}{dt}\left(g_i(t)\right)g_i(t)^{-1}e_\alpha\right)\right] \\
&= [(\sigma_i(p), \mathcal{A}_i(X)e_\alpha)]
\end{aligned} \tag{205}$$

where we used the ODE (164) for $g_i(t)$. Hence, in terms of the basis $\{s_\alpha\}$,

$$\nabla_X s_\alpha = \mathcal{A}_i(X)s_\alpha \equiv (\mathcal{A}_i(X))^\beta{}_\alpha s_\beta \tag{206}$$

□

The covariant derivative of s_α's determines all sections' covariant derivatives, by (CD2).

Corollary C.6. *Let ξ be a smooth section on U_i such that $\xi = \xi^\alpha(p)s_\alpha$.*

$$\begin{aligned}
\nabla_X\xi &= \xi^\alpha(\mathcal{A}_i(X))^\beta{}_\alpha s_\beta + (X\xi^\alpha)s_\alpha \\
&= X^\mu\left(\xi^\beta\mathcal{A}_{i\mu}{}^\alpha{}_\beta + \frac{\partial\xi^\alpha}{\partial x^\mu}\right)s_\alpha
\end{aligned} \tag{207}$$

In practice, we can choose an arbitrary frame $\{s_\alpha\}$. A frame provides a local trivialization: $\phi : U \times V \to \pi^{-1}(U)$, such that $\phi(p, e_\alpha) \mapsto s_\alpha(p)$. By (207). We can define,

$$\nabla s_\alpha \equiv \mathcal{A}^\beta{}_\alpha s'_\beta. \tag{208}$$

For another frame $s'_\alpha = M(p)^\beta{}_\alpha s_\beta$, $M(p) \in GL(V, \mathbb{F})$, the new connection matrix is,

$$\mathcal{A}'^{\beta}_{i\ \alpha} = (M^{-1})^\beta{}_\gamma (\mathcal{A})^\gamma{}_\delta M^\delta{}_\alpha + (M^{-1})^\beta{}_\gamma dM^\gamma{}_\alpha \tag{209}$$

In particular, if $\{s_\alpha\}$ and $\{s'_\alpha\}$ are the canonical sections on U_i and U_j respectively, then

$$M^\beta{}_\alpha = (t_{ij})^\beta{}_\alpha, \tag{210}$$

and the transition (209) is the V-representation of the connection transition law on the principal bundle.

Example C.7 (frame bundle and tangent bundle).

C.5 Local curvature form

Let $\mathcal{F}$ be the local curvature form on M for the principal bundle. The local curvature form for the associated bundle is defined to be $\mathcal{F}^\alpha{}_\beta$, the V-representation of F. In components,

$$\mathcal{F}^\beta{}_\alpha = d\mathcal{A}^\beta{}_\alpha + \mathcal{A}^\beta{}_\gamma \wedge \mathcal{A}^\gamma{}_\alpha. \tag{211}$$

If we change the sections as $s'_\alpha = M(p)^\beta{}_\alpha s_\beta$, then,

$$\mathcal{F}'^\beta{}_\alpha = (M^{-1})^\beta{}_\gamma (\mathcal{F})^\gamma{}_\delta M^\delta{}_\alpha. \tag{212}$$

where $\mathcal{F}'^\beta{}_\alpha$ is calculated from $\mathcal{A}'^\beta{}_\alpha$.

The local curvature form is determined by the covariant derivatives.

Proposition C.8. *For two vector fields X, Y on M and a smooth section s for E,*

$$(\nabla_X \nabla_Y - \nabla_Y \nabla_X) s_\alpha = \mathcal{F}^\beta{}_\alpha(X, Y) s_\beta + \nabla_{[X,Y]} s_\alpha. \tag{213}$$

Proof. By (C.5) and (CD2), we have

$$\begin{aligned}
\nabla_X \nabla_Y s_\alpha &= \nabla_X (\mathcal{A}^\beta{}_\alpha(Y) s_\beta) = \mathcal{A}^\gamma{}_\beta(X) \mathcal{A}^\beta{}_\alpha(Y) s_\gamma + X(\mathcal{A}^\beta{}_\alpha(Y)) s_\beta, \\
\nabla_Y \nabla_X s_\alpha &= \nabla_Y (\mathcal{A}^\beta{}_\alpha(X) s_\beta) = \mathcal{A}^\gamma{}_\beta(Y) \mathcal{A}^\beta{}_\alpha(X) s_\gamma + Y(\mathcal{A}^\beta{}_\alpha(X)) s_\beta
\end{aligned} \tag{214}$$

Note that,

$$d\mathcal{A}^\beta{}_\alpha(X, Y) = X(\mathcal{A}^\beta{}_\alpha(Y)) - Y(\mathcal{A}^\beta{}_\alpha(X)) - \mathcal{A}^\beta{}_\alpha([X, Y]), \tag{215}$$

so (213) holds. $\square$

Formally, we can extend ∇ as a linear map $\Gamma(M, E\otimes\Lambda^p(M^*)) \to \Gamma(M, E\otimes\Lambda^{p+1}(M^*))$,

$$\nabla(s\otimes\eta) \equiv (\nabla s)\wedge\eta + s\otimes d\eta \tag{216}$$

for $s\in\Gamma(M,E)$, $\eta\in\Omega^p(M)$. Then it is straightforward to check that,

$$\nabla\nabla s_\alpha = s_\beta\otimes\mathcal{F}^\beta{}_\alpha \tag{217}$$

Unlike the covariant derivative, $\mathcal{F}(X,Y):\pi^{-1}(p)\to\pi^{-1}(p)$ is well-defined,

$$\mathcal{F}(X,Y)(c^\alpha s_\alpha(p)) \equiv s_\beta(p)\mathcal{F}^\beta{}_\alpha(X,Y)c^\alpha, \tag{218}$$

and it is straightforward to check that it is independent of the frame choice. If the vector bundle is associated with the principle bundles P, using the canonical trivialization (U_i,σ_i), we get

$$\mathcal{F}(X,Y)([(\sigma_i(p),s_\alpha)]) = [(\sigma_i(p), s_\beta\mathcal{F}^\beta{}_\alpha(X,Y))], \tag{219}$$

where $\mathcal{F}^\beta{}_\alpha$ is the V-representation matrix of $\mathcal{F}_i$, the curvature form on the principal bundle. Hence in $Gl(\pi^{-1}(p),\mathbb{F})$,

$$\mathcal{F}(X,Y) = L_{\sigma_i(p)}(\mathcal{F}_i(X,Y)) \tag{220}$$

where we use the same notation $L_{\sigma_i(p)}$ for the Lie algebra map.[7]

Theorem C.9 (Ambrose-Singer)**.** *Hol_p's Lie algebra is spanned by $\Gamma_{\gamma^{-1}}\cdot\mathcal{F}(X,Y)\cdot\Gamma_\gamma$, where γ is piecewise smooth curve in M with $\gamma(0)=p$ and $\gamma(1)=q$. $X,Y\in T_qM$.*

Proof. We have following diagram,

$$\begin{tikzcd} G \arrow[r, "L_{\phi_i(p)}"] \arrow[d, dash] & Gl(\pi^{-1}(p),\mathbb{F}) \arrow[d, dash] \\ \Phi_{\sigma_i(p)} \arrow[r, "L_{\phi_i(p)}"] & \mathrm{Hol}_p \end{tikzcd} \tag{221}$$

where the section line is surjective. By the Ambrose-Singer theorem for principal bundles, $\Phi_{\sigma_i(p)}$'s Lie algebra is spanned by $\{\Omega(X,Y)|X,Y\in H_vP\}$ for all $u'\in P$ which can be connected to $\sigma_i(p)$ by a horizontal curve. Without

[7] Explicitly, we can verify that the definition is independent of the local section choice. $L_{\sigma_j(p)}(\mathcal{F}_j) = L_{\sigma_i(p)t_{ij}(p)}(\mathcal{F}_j) = L_{\sigma_i(p)}(t_{ij}(p))L_{\sigma_i(p)}(\mathcal{F}_j)L_{\sigma_i(p)}(t_{ij}^{-1}(p)) = L_{\sigma_i(p)}(\mathcal{F}_i)$.

loss of generality, we defined $\tilde{\gamma}$ be such a curve with $\tilde{\gamma}(0) = \sigma_i(p)$ and $\tilde{\gamma}(1) = \sigma_j(q)$, $q \in U_j \subset M$. Let $\gamma(t) = \pi(\tilde{\gamma}(t))$. It is clear that,

$$L_{\sigma_i(p)}(\Omega(\tilde{X}, \tilde{Y}))[(\sigma_i(p), v)] = [(\sigma_i(p), \Omega(\tilde{X}, \tilde{Y})v)] \tag{222}$$

for $\tilde{X}, \tilde{Y} \in H_{\sigma_j(q)}P$. On the other hand, we can find vectors $X, Y \in T_qM$, such that,

$$\Omega(\tilde{X}, \tilde{Y}) = \Omega(\sigma_{j*}(X), \sigma_{j*}(Y)) = \mathcal{F}_j(X, Y). \tag{223}$$

Furthermore, the parallel transport of γ induced a linear isomorphism $\pi_E^{-1}(p) \to \pi_E^{-1}(q)$,

$$\gamma : [(\sigma_i(p), v)] \mapsto [(\sigma_j(q), v)] \tag{224}$$

Hence,

$$\begin{aligned} L_{\sigma_i(p)}(\Omega(\tilde{X}, \tilde{Y}))[(\sigma_i(p), v)] &= [(\sigma_i(p), \mathcal{F}_j(X, Y)v)] \\ &= \gamma^{-1} \cdot L_{\sigma_j(q)}(\mathcal{F}_j(X, Y)) \cdot \gamma[(\sigma_i(p), v)] \\ &= \gamma^{-1} \cdot \mathcal{F}(X, Y) \cdot \gamma[(\sigma_i(p), v)] \end{aligned} \tag{225}$$

Hence $L_{\sigma_i(p)}(\Omega(\tilde{X}, \tilde{Y})) = \gamma^{-1} \cdot \mathcal{F}(X, Y) \cdot \gamma$. □

Definition C.10. We define the *Ricci form* to be the trace of $\mathcal{F}$,

$$\mathfrak{R} = \mathcal{F}^{\alpha}{}_{\alpha} \tag{226}$$

which is frame independent. A connection is called *Ricci flat* if $\mathfrak{R} \equiv 0$.

Corollary C.11. *If the connection is Ricci flat, then* $\forall p \in U_i$, $\mathrm{Hol}_p \subset sl(\pi^{-1}(p), \mathbb{F})$.

Proof. The Lie algebra of Hol_p is $\{\mathcal{F}(X, Y) | X, Y \in T_pM\}$.

$$tr(\mathcal{F}(X, Y)) = \mathcal{F}^{\alpha}{}_{\alpha}(X, Y) = 0. \tag{227}$$

□

C.6 Riemann structure, revisited

In this section, we revisit our old friend, Riemann structure. Our discussion work for any real vector bundle $E \to M$, not restricted to the tangent bundle TM.

A *Riemann structure* on E means there is continuously defined inner product for each fibre $\pi^{-1}(p)$, $p \in M$.

$$g_p : \pi^{-1}(p) \times \pi^{-1}(p) \to \mathbb{R} \tag{228}$$

A connection ∇ is called a *metric connection* if it preserves the inner product: for two sections s_1, s_2 and a vector $X \in T_pM$

$$X(g(s_1, s_2)) = g(\nabla_X s_1, s_2) + g(s_1, \nabla_X s_2). \tag{229}$$

In the other word, $\delta_X g = 0$ if we extend the covariant derivative to the dual space of V. The metric connection is restricted by the Riemann structure: Let $g(s_\alpha, s_\beta) = g_{\alpha\beta}$, we have

$$dg_{\alpha\beta} = g_{\gamma\beta}\mathcal{A}^\gamma{}_\alpha + g_{\alpha\gamma}\mathcal{A}^\gamma{}_\beta \quad , dg = g\mathcal{A} + \mathcal{A}^T g \tag{230}$$

Take the exterior derivative, we have,

$$\mathcal{F}^\beta{}_\alpha = -g_{\alpha\gamma}\mathcal{F}^\gamma{}_\delta g^{\delta\beta}, \quad \mathcal{F}^T = -g\mathcal{F}g^{-1} \tag{231}$$

Example C.12. If $E = TM$, we can choose $U_i = \{(x_1, ..., x_n)\}$ and the sections $s_\mu = \partial/\partial x_\mu$. In this case, (230) and the torsion free condition

$$\mathcal{A}^\mu{}_{\nu\lambda} = \mathcal{A}^\mu{}_{\lambda\nu}, \quad \mathcal{A}^\mu{}_\lambda = \mathcal{A}^\mu{}_{\nu\lambda} dx^\nu \tag{232}$$

uniquely determine the *Levi-Civita connection*, $\Gamma^\mu_{\nu\lambda} \equiv \mathcal{A}^\mu{}_{\nu\lambda}$. Furthermore, the *Riemann curvature tensor* is defined by,

$$R^\rho{}_{\sigma\mu\nu} \equiv (\mathcal{F}^\rho{}_\sigma)_{\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}. \tag{233}$$

However, when $E \neq TM$, there may not exist a natural way to define the torsion-free condition or the corresponding Levi-Civita connection.

Locally, we can always choose an orthonormal frame $\{\hat{e}_\alpha\}$ for E such that $g(\hat{e}_\alpha, \hat{e}_\beta) = \delta_{\alpha\beta}$. The connection form for $\{\hat{e}_i\}$ satisfy,

$$\mathcal{A}^\beta{}_\alpha = -\mathcal{A}^\alpha{}_\beta \tag{234}$$

which is $\mathfrak{so}$ Lie-algebra-valued local connection one forms. This implies that the structure group $G = GL(\mathbb{R})$ can be reduced to $O(\mathbb{R})$.

C.7 Holomorphic vector bundle and Hermitian structure

In this section, we follow both [1] and [4]. Let $\pi : E \to M$ be a *complex vector bundle*. The fibre has complex dimension k. The manifold E is a *holomophic vector bundle* if its a complex vector bundle and satisfy,

1. E and M are complex manifold and π is a holomorphic map.
2. The local trivialization $\phi_i : U_i \times \mathbb{C} \to \pi^{-1}(U_i)$ is a biholomorphism.
3. The transition function $t_{ij} : U_i \cap U_j \to GL(k, \mathbb{C})$ is holomorphic.

Example C.13. Let M be a complex manifold with $dim_{\mathbb{C}} = m$. For $p \in M$, we choose p's coordinate neighborhood $U_i = \{z^1, ...z^m\}$. The *holomorphic tangent space* at p is

$$TM_p^+ \equiv T^{(1,0)}M = \mathrm{span}_{\mathbb{C}}\{\frac{\partial}{\partial z^1}, ..., \frac{\partial}{\partial z^m}\}. \tag{235}$$

By the complex structure of M, TM_p^+ is independent of the coordinate choice. We define the holomorphic tangent bundle as

$$TM = \bigcup_{p \in M} T_pM^+, \tag{236}$$

which is a $2m$-dimensional complex manifold with complex structure defined by $\{z^1, ..., z^m, c^1, ..., c^m\}$, where c^μ is the complex coefficient of $\partial/\partial z^\mu$. It clear that for another coordinate neighborhood $U_j = \{w^1, ..., w^m\}$,

$$(t_{ij})^\mu{}_\nu = \frac{\partial z^\mu}{\partial w^\nu}, \tag{237}$$

which is a holomorphic map from M to $GL(m, \mathbb{C})$. So TM^+ is a holomorphic vector bundle.

For a section $s \in \Gamma(M, E)$, we can define $\bar{\partial} s \in \Gamma(M, E \otimes (T^*M)^{\mathbb{C}})$. For $\forall p \in U_i$, choose a map $s_i : U_i \to \mathbb{C}^k$ such that,

$$\phi_i(p, s_i(p)) = s \tag{238}$$

Then for a vector $X \in T_pM^{\mathbb{C}}$,

$$\bar{\partial} s(X) = \phi_i(p, \bar{\partial} s_i(X)). \tag{239}$$

$\bar{\partial}s$ is well-defined because if we use a different trivialization,

$$\phi_j(p, s_j(p)) = s, \quad s_j(p) = t_{ji}(p)s_i(p) \tag{240}$$

then

$$\phi_j(p, \bar{\partial}s_j(X)) = \phi_j(p, (\bar{\partial}t_{ji}(p)s_i)(X)) = \phi_j(p, t_{ji}(p)\bar{\partial}s_i(X)), \tag{241}$$

because $\bar{\partial}t_{ji} = 0$. Note that there is no natural way to define ∂s. Formally, we can extend $\bar{\partial}$ as linear map $\Gamma(M, E\otimes\Lambda^p(M^*)^{\mathbb{C}}) \to \Gamma(M, E\otimes\Lambda^{p+1}(M^*)^{\mathbb{C}})$,

$$\bar{\partial}(s\otimes\eta) \equiv (\bar{\partial}s)\wedge\eta + s\otimes\bar{\partial}\eta \tag{242}$$

for $s \in \Gamma(M, E)$, $\eta \in \Omega^p(M)^{\mathbb{C}}$. Again,

Proposition C.14. $\bar{\partial}\bar{\partial} = 0$.

Proof. Introduce a local coordinate neighborhood $[(z^1, ..., z^n)]$, and define $\bar{\partial}s \equiv \bar{\partial}_\mu s \otimes d\bar{z}^\mu$. Each s_μ is a local section in E.

$$\begin{aligned} \bar{\partial}(s\otimes\eta) &= \bar{\partial}(s_\mu \otimes (d\bar{z}^\mu \wedge \eta) + s\otimes\bar{\partial}\eta) \\ &= s_{\mu\nu}(d\bar{z}^\nu \wedge d\bar{z}^\mu \wedge \eta) = 0. \end{aligned} \tag{243}$$

where we defined $\bar{\partial}s_\mu \equiv s_{\mu\nu}\otimes d\bar{z}^\mu$ and used the fact $s_{\mu\nu} = s_{\nu\mu}$, $\bar{\partial}\bar{\partial}\eta = 0$. □

Definition C.15 (Hermitian structure)**.** For a holomorphic vector bundle E, a Hermitian structure is the Hermitian inner products h defined $\forall p \in M$, which satisfies,

1. $h_p(c_1u_1 + c_2u_2, v) = c_1h_p(u_1, v) + c_2h_p(u_2, v)$, $c_1, c_2 \in \mathbb{C}$ and $u_1, u_2, v \in \pi^{-1}(p)$, [8]
2. $h_p(u, v) = \overline{h_p(v, u)}$,
3. $h_p(u, u) \geq 0$, while the equality holds only if $u = 0 \in \pi^{-1}(p)$,
4. $h(s_1, s_2)$ is a complex smooth function on M, if $s_1, s_2 \in \Gamma(M, E)$.

[8]This convention follows [4] but is different from [1].

For a frame s_α, we define $h_{\alpha\bar{\beta}} = h(s_\alpha, s_\beta)$. So the inner products for $s_1 = f^\alpha s_\alpha$ and $s_2 = g^\alpha s_\alpha$,

$$h(s_1, s_2) = h_{\alpha\bar{\beta}} f^\alpha \bar{g}^\beta. \tag{244}$$

We can formally define the *Hermitian connection* on a holomorphic vector bundle E,

Definition C.16. The Hermitian connection ∇ is a $\mathbb{C}$-linear map $\Gamma(M, E) \to \Gamma(M, E \otimes (T^*M)^{\mathbb{C}})$, which satisfies:

HC1 $\nabla(fs) = (df)s + f\nabla s$, where f is a complex smooth function on M and $s \in \Gamma(M, E)$,

HC2 $d[h(s_1, s_2)] = h(\nabla s_1, s_2) + h(s_1, \nabla s_2)$,

HC3 $\nabla s = Ds + \bar{D}s$, where $Ds \in \Gamma(M, E \otimes T^{(1,0)}M)$ and $\bar{D}s \in \Gamma(M, E \otimes T^{(0,1)}M)$. We require that $\bar{D}s = \bar{\partial}s$.

The Hermitian connection uniquely exists for a Hermitian structure. Again, we can extend ∇ and D as linear map $\Gamma(M, E \otimes \Lambda^p(M^*)^{\mathbb{C}}) \to \Gamma(M, E \otimes \Lambda^{p+1}(M^*)^{\mathbb{C}})$,

$$\begin{aligned} \nabla(s \otimes \eta) &\equiv (\nabla s) \wedge \eta + s \otimes d\eta, \\ D(s \otimes \eta) &\equiv (Ds) \wedge \eta + s \otimes \partial\eta, \end{aligned} \tag{245}$$

for $s \in \Gamma(M, E)$, $\eta \in \Omega^p(M)^{\mathbb{C}}$.

As before, let $\nabla s_\alpha = s_\beta \mathcal{A}^\beta{}_\alpha$. (HC2) can be written as,

$$dh_{\alpha\bar{\beta}} = h_{\gamma\bar{\beta}} \mathcal{A}^\gamma{}_\alpha + h_{\alpha\bar{\gamma}} \overline{\mathcal{A}^\gamma{}_\beta}, \quad \text{or } dh = \mathcal{A}^T h + h\overline{\mathcal{A}} \tag{246}$$

Take the exterior derivative, we have,

$$\overline{\mathcal{F}^\alpha{}_\beta} = -h^{\bar{\alpha}\gamma} \mathcal{F}^\delta{}_\gamma h_{\delta\bar{\beta}}, \quad \text{or } \overline{\mathcal{F}} = -h^{-1}\mathcal{F}^T h \tag{247}$$

where $h^{\bar{\alpha}\beta}$ is the inverse of the metric matrix such that $h^{\bar{\alpha}\beta} h_{\beta\bar{\gamma}} = \delta^\delta{}_\gamma$.

We can choose the local *holomorphic frame* s_α for E. A local section s is called *holomorphic*, if and only if, under local trivialization $\phi_i : U_i \times \mathbb{C}^k \to \pi^{-1}(U_i)$,

$$\phi_i(p, s_i(p)) = s \tag{248}$$

$s_i(p)$ is a holomorphic map $U_i \to \mathbb{C}^k$. This definition is independent of the U_i choice. It is clear that $\bar{\partial}s = 0$ for a holomorphic section. Therefore, $\mathcal{A}^\alpha{}_\beta$ for the holomorphic frame is a $(1, 0)$-form. Furthermore, by (246), extract the holomorphic part and we have

Theorem C.17. *For the holomorphic frame* $\mathcal{A}^{\alpha}{}_{\beta} = h^{\bar{\gamma}\alpha}\partial h_{\beta\bar{\gamma}}$*, or* $\mathcal{A} = (h^{-1})^T\partial h^T$*. Hence the Hermitian connection for a Hermitian structure uniquely exists.*

Corollary C.18. *For the holomorphic frame, the curvature form* $\mathcal{F}^{\alpha}{}_{\beta} = \bar{\partial}\mathcal{A}^{\alpha}{}_{\beta}$ *and is a* $(1,1)$*-form.*

Proof.

$$\begin{aligned}
\mathcal{F} &= d\mathcal{A} + \mathcal{A}\wedge\mathcal{A} \\
&= -(h^{-1}dhh^{-1})^T \wedge \partial h^T + (h^{-1})^T\partial\bar{\partial}h^T + (h^{-1})^T\partial h^T \wedge (h^{-1})^T\partial h^T \\
&= -(h^{-1})^T\bar{\partial}h^T \wedge (h^{-1})^T\partial h^T + (h^{-1})^T\partial\bar{\partial}h^T \\
&= \bar{\partial}\mathcal{A} \qquad (249)
\end{aligned}$$

So $\mathcal{F}^{\alpha}{}_{\beta}$ is a $(1,1)$-form. □

The trace of $\mathcal{F}$ is,

$$tr\mathcal{F} = \mathcal{F}^{\alpha}{}_{\alpha} = \bar{\partial}\partial(\log\det(h)). \qquad (250)$$

Alternatively, we can locally choose the orthonormal frame $\{\hat{e}_\alpha\}$ such that E such that $g(\hat{e}_\alpha, \hat{e}_\beta) = \delta_{\alpha\beta}$. Let $\nabla\hat{e}_\alpha = \hat{e}_\beta\mathcal{A}^{\beta}{}_{\alpha}$. Then, as before $\mathcal{F}^{\beta}{}_{\alpha} = d\mathcal{A}^{\beta}{}_{\alpha} + \mathcal{A}^{\beta}{}_{\gamma}\wedge\mathcal{A}^{\gamma}{}_{\alpha}$. From (HC2),

$$\mathcal{A}^{\beta}{}_{\alpha} = -\overline{\mathcal{A}^{\alpha}{}_{\beta}}, \quad \mathcal{F}^{\beta}{}_{\alpha} = -\overline{\mathcal{F}^{\alpha}{}_{\beta}} \qquad (251)$$

Proposition C.19. *The curvature form* $\mathcal{F}^{\beta}{}_{\alpha}$ *for an orthonormal frame* $\{\hat{e}_\alpha\}$ *is a* $(1,1)$*-form.*

Proof.

$$\nabla\nabla\hat{e}_\alpha = DD\hat{e}_\alpha + D\bar{\partial}\hat{e}_\alpha + \bar{\partial}D\hat{e}_\alpha = \mathcal{F}^{\beta}{}_{\alpha}\hat{e}_\beta \qquad (252)$$

By (245), DDe_α is $(2,0)$-form-valued section while $D\bar{\partial}e_\alpha$ and $\bar{\partial}De_\alpha$ are $(1,1)$-form-valued sections. Hence $\mathcal{F}^{\beta}{}_{\alpha}$ has no $(0,2)$ component. However, by (251), $\mathcal{F}^{\beta}{}_{\alpha}$ has no $(2,0)$ component either. □

C.8 Complex geometry

Following the analysis and example, we can quickly develop the basic concepts about complex geometry.

Suppose the holomorphic tangent bundle TM^+ of a complex manifold M has a Hermitian metric h. $\dim_{\mathbb{C}} M = m$. In the local coordinate $(z_1, ..., z_m)$,

$$h(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}) \equiv h_{\mu\bar{\nu}} \tag{253}$$

$h_{\mu\bar{\nu}}$ is a positive-definite Hermitian matrix. In another coordinate patch $(w_1, ..., w_2)$, the Hermitian metric reads,

$$h(\frac{\partial}{\partial w^\mu}, \frac{\partial}{\partial w^\nu}) \equiv h'_{\mu\bar{\nu}} = h_{\lambda\bar{\rho}} \frac{\partial z^\lambda}{\partial w^\mu} \overline{\left(\frac{\partial z^\rho}{\partial w^\nu}\right)} \tag{254}$$

The inverse of $h_{\mu\bar{\nu}}$ is $h^{\bar{\lambda}\rho}$ such that $h^{\bar{\lambda}\rho} h_{\rho\bar{\nu}} = \delta^{\bar{\lambda}}{}_{\bar{\nu}}$.

Simply, we define the conjugate Hermitian metric $\bar{h}$ on the *anti-holomorphic-bundle* as,

$$\bar{h}(\frac{\partial}{\partial \bar{z}^\mu}, \frac{\partial}{\partial \bar{z}^\nu}) \equiv h_{\bar{\mu}\nu} = \overline{h_{\mu\bar{\nu}}} \tag{255}$$

all the analysis on $\bar{h}$ is identical to h.

h on TM^+ and $\bar{h}$ on TM^- naturally defined a complex-bilinear function g on the whole tangent space. For $X, Y \in (T_p M)^{\mathbb{C}}$, we decompose them to the holomorphic and anti-holomorphic parts, $X = X^H + X^A$, $Y = Y^H + Y^A$,

$$g(X, Y) \equiv h(X^H, \overline{Y^A}) + \bar{h}(X^A, \overline{Y^H}) \tag{256}$$

such that $g(\overline{X}, \overline{Y}) = \overline{g(X, Y)}$, $g(X, Y) = g(Y, X)$. Locally, in terms of the one-forms, g is

$$g = h_{\mu\bar{\nu}} dz^\mu \otimes d\bar{z}^\nu + h_{\bar{\mu}\nu} d\bar{z}^\mu \otimes dz^\nu = h_{\mu\bar{\nu}}(dz^\mu \otimes d\bar{z}^\nu + d\bar{z}^\nu \otimes dz^\mu) \tag{257}$$

Restrict g to the real tangent space directly, we get a Riemann metric on M, as a real manifold.

$$ds_R^2 = 2Re(h_{\mu\bar{\nu}})(dx^\mu dx^\nu + dy^\mu dy^\nu) + 2Im(h_{\mu\bar{\nu}})(dx^\mu dy^\nu + dy^\nu dx^\mu) \tag{258}$$

The metric-compatible holomorphic connection for TM^+ uniquely exists, which reads,

$$\nabla_\mu \left(\frac{\partial}{\partial z^\nu}\right) = \Gamma^\lambda_{\mu\nu} \cdot \left(\frac{\partial}{\partial z^\lambda}\right), \quad \nabla_{\bar{\mu}} \left(\frac{\partial}{\partial z^\nu}\right) = 0 \tag{259}$$

Simply, we can define the metric-compatible anti-holomorphic connection for TM^-,

$$\nabla_{\bar{\mu}}\left(\frac{\partial}{\partial z^{\bar{\nu}}}\right) = \Gamma^{\bar{\lambda}}_{\bar{\mu}\bar{\nu}} \cdot \left(\frac{\partial}{\partial \bar{z}^{\lambda}}\right), \quad \nabla_{\mu}\left(\frac{\partial}{\partial z^{\bar{\nu}}}\right) = 0 \tag{260}$$

So the only non-vanishing components of Γ^A_{BC} are $\Gamma^{\lambda}_{\mu\nu}$ and $\Gamma^{\bar{\lambda}}_{\bar{\mu}\bar{\nu}}$.

The connection matrix $\Gamma^{\lambda}_{\mu\nu}$ can easily calculated by theorem (C.17) as $\Gamma = (h^{-1})^T \partial h^T$,

$$\Gamma^{\lambda}_{\mu\nu} = h^{\bar{\gamma}\lambda}\partial_{\mu} h_{\nu\bar{\gamma}}, \tag{261}$$

and similarly,

$$\Gamma^{\bar{\lambda}}_{\bar{\mu}\bar{\nu}} = \overline{\Gamma^{\lambda}_{\mu\nu}}. \tag{262}$$

The curvature tensor is calculated by the simple formula (C.18), $R = \bar{\partial}\Gamma$,

$$R^{\lambda}{}_{\nu\bar{\rho}\mu} = \partial_{\bar{\rho}}\Gamma^{\lambda}_{\mu\nu} \tag{263}$$

which is a matrix-valued $(1,1)$-form in $\bar{\rho}, \mu$. So $R^{\lambda}{}_{\nu\bar{\rho}\mu} = -R^{\lambda}{}_{\nu\mu\bar{\rho}}$. Similarly,

$$R^{\bar{\lambda}}{}_{\bar{\nu}\rho\bar{\mu}} = \overline{R^{\lambda}{}_{\nu\bar{\rho}\mu}} \tag{264}$$

All the other components of $R^A{}_{BCD}$ vanish.

In particular, if $m = 1$, then the metric matrix is a positive function h and the curvature tensor is a $(1,1)$ form

$$R = \bar{\partial}h^{-1}\partial h = \bar{\partial}\partial\log(h) = -\partial\bar{\partial}\log(h) \tag{265}$$

The Ricci tensor is defined to be the trace over the matrix indices. From (250)

$$\mathfrak{R} = \mathfrak{R}_{\mu\bar{\rho}} dz^{\mu} \wedge d\bar{z}^{\rho} = \sum_{\lambda} R^{\lambda}{}_{\lambda\mu\bar{\rho}} dz^{\mu} \wedge d\bar{z}^{\rho} = \bar{\partial}\partial(\log\det(h)) \tag{266}$$

Then by the definition of the first Chern classes,

$$c_1(TM^+) = \frac{i}{2\pi}\mathfrak{R} \tag{267}$$

which is a real and closed 2-form.

So far, the *torsion-free* condition is not imposed and actually in general,

$$\Gamma^{\lambda}_{\mu\nu} \neq \Gamma^{\lambda}_{\nu\mu}, \tag{268}$$

There is a particular class of complex manifold, Kähler manifolds, on which the *torsion-free* condition automatically holds,

Define the $(1,1)$-form associated with h,

$$\omega = i \cdot h_{\mu\bar{\nu}} dz^\mu \wedge d\bar{z}^\nu. \tag{269}$$

Note that

$$\overline{\omega} = -ih_{\bar{\mu}\nu} d\bar{z}^\mu \wedge dz^\nu = ih_{\nu\bar{\mu}} d\bar{z}^\nu \wedge d\bar{z}^\mu = \omega \tag{270}$$

so ω is a real form.

Definition C.20. A complex manifold equipped with the Hermitian metric h is a Kähler manifold, if and only if $d\omega = 0$.

Theorem C.21. *Locally, there exists a smooth function $\mathcal{K}$ such that*

$$h_{\mu\bar{\nu}} = \partial_\mu \bar{\partial}_\nu \mathcal{K} \tag{271}$$

$\mathcal{K}$ is called the Kähler potential.

Theorem C.22. *A Kähler manifold is torsion-free, $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$*

The advantage of Kähler manifolds is that they are largely similar to algebraic varieties.

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